

# WEIERSTRASS EQUATIONS FOR JACOBIAN FIBRATIONS ON A CERTAIN $K3$ SURFACE

KAZUKI UTSUMI

ABSTRACT. In this paper we give the Weierstrass equations for Jacobian fibrations on the  $K3$  surface that is the minimal resolution of the double covering of  $\mathbb{P}^2$  ramified along generic six lines.

## 1. INTRODUCTION

**1.1. Problem setting.** Let  $X$  be a  $K3$  surface defined over an algebraically closed field  $k$  with  $\text{char}(k) \neq 2, 3$ . Suppose  $f : X \rightarrow \mathbb{P}^1$  is a Jacobian fibration, that is, an elliptic fibration on  $X$  with a section  $O : \mathbb{P}^1 \rightarrow X$ . Let  $t$  be an affine coordinate of  $\mathbb{P}^1$ . Then  $f^*(t)$  defines a non-constant rational function on  $X$ , which is called an *elliptic parameter* for the Jacobian fibration  $f$ . We also denote  $f^*(t)$  by  $t$  and regard  $t$  as a rational function on  $X$ . The generic fiber of  $f$  defines an elliptic curve  $E$  over the rational function field  $k(t)$ .

Kuwata and Shioda [7] proposed the following problems.

**Problem 1.** *Given a  $K3$  surface  $X/k$  and a Jacobian fibration  $f$ , determine (i) the elliptic parameter  $t$  for  $f$ , (ii) the defining equation of the elliptic curve  $E/k(t)$ , and (iii) the Mordell-Weil lattice (MWL)  $E(k(t))$ .*

**Problem 2.** *Given a  $K3$  surface  $X/k$ , determine all the (essentially distinct) elliptic parameters.*

Problem 2 is a combination of Problem 1 and the following standard problem:

**Problem 3.** *Given a  $K3$  surface  $X/k$ , classify the Jacobian fibration  $f : X \rightarrow \mathbb{P}^1$  up to isomorphism.*

Oguiso [8] solved Problem 3 in the case where  $X$  is a Kummer surface of the product of non-isogenous elliptic curves over the complex number field. Namely he classified the configuration of singular fibers on such a Kummer surface  $X$  into eleven types  $\mathcal{J}_1, \dots, \mathcal{J}_{11}$ , and determined the number of the isomorphism classes for each type.

Kuwata and Shioda [7] solved Problems 1 and 2 for each member of Oguiso's list. They gave elliptic parameters and Weierstrass equations for each member of Oguiso's list by using the Legendre parameters of the two elliptic curves.

Recently, Kumar [5] solved the above problems completely for the case of a generic Jacobian Kummer surface  $X = \text{Km}(J(C))$ , for a genus 2 curve  $C$  over an algebraically closed field of characteristic 0. He showed that there are exactly 25 distinct Jacobian fibrations on such a generic Kummer surface, and gave an elliptic parameter, the Weierstrass equation and the Mordell-Weil lattice for each type.

---

2010 *Mathematics Subject Classification.* 14J27, 14J28, 14H52.

*Key words and phrases.*  $K3$  surface, elliptic surface, elliptic curve.

**1.2. Main results.** In this paper, we focus on the case of a  $K3$  surface over  $k$  that is the minimal resolution of the double covering of  $\mathbb{P}^2$  ramified along *generic* six lines. When we say that six lines are *generic*, we mean that the rank of the Néron-Severi group  $NS(X)$  is 16.

In this case, Problem 3 has been partially solved by Kloosterman [3]. He classified all configurations of singular fibers of Jacobian fibrations on such a  $K3$  surface over a field of characteristic 0 (although most of his results hold over fields of any characteristic, not 2 or 3) into sixteen classes. However he did not give the classification of the isomorphism classes for each type. Thus, this gives a partial solution to Problem 3 on such a  $K3$  surface.

Table 1 shows a summary of Kloosterman's results in the generic case. The first column shows the class of Jacobian fibration following Kloosterman's notation. The second column shows the configuration of singular fibers. Here, for example, by  $I_2^* + 8I_2$  we mean that the surface has two fibers of type  $I_2^*$  (Kodaira's notation [4]) and eight fibers of type  $I_2$ . The third column shows the Mordell-Weil group (MWG) of the fibration.

Our main results are as follows: we solve (i) and (ii) of Problem 1 for each class of Table 1. We give an elliptic parameter and its Weierstrass equation for one Jacobian fibration in each class of the table, although there may exist nonisomorphic Jacobian fibrations belonging to the same class of the list. More details will be given in § 1.3 after we fix the notation.

**1.3. Notation.** Fix generic six lines  $L_i \subset \mathbb{P}^2$ . Denote by  $P_{i,j}$  the point of intersection of  $L_i$  and  $L_j$ .

Let  $\varphi' : Y \rightarrow \mathbb{P}^2$  be the double cover ramified along the six lines  $L_i$ . Then  $Y$  has 15 double points of type  $A_1$ , which correspond to  $P_{i,j}$ . Blowing up these points gives a  $K3$  surface  $X$ , with 15 exceptional divisors  $\ell_{i,j}$  and a rational map  $\varphi : X \rightarrow \mathbb{P}^2$ . For a curve  $C$  on  $\mathbb{P}^2$ , we call the strict transform of  $\varphi'^*(C)$  the *pull-back of  $C$* , for short. Let  $\ell_i$  be the divisor on  $X$  such that  $2\ell_i$  is the pull-back of  $L_i$ . Let  $\mu_{k,m}^{i,j}$  be the pull-back of the line  $M_{k,m}^{i,j}$  connecting  $P_{i,j}$  and  $P_{k,m}$  with  $i, j, k, m$  pairwise distinct. With this notation, which is the same as Kloosterman [3], divisors  $\ell_i, \ell_{i,j}, \mu_{k,m}^{i,j}$  are  $(-2)$ -curves on  $X$ . We have the following intersection numbers.

$$\begin{aligned}
 \ell_i \cdot \ell_j &= \begin{cases} -2 & i = j \\ 0 & i \neq j \end{cases}, \quad \ell_{i,j} \cdot \ell_{k,m} = \begin{cases} -2 & \{i, j\} = \{k, m\} \\ 0 & \text{otherwise} \end{cases}, \\
 \ell_i \cdot \ell_{k,m} &= \begin{cases} 1 & i \in \{k, m\} \\ 0 & \text{otherwise} \end{cases}, \quad \ell_p \cdot \mu_{k,m}^{i,j} = \begin{cases} 1 & p \notin \{i, j, k, m\} \\ 0 & \text{otherwise} \end{cases}, \\
 \ell_{i,j} \cdot \mu_{p,q}^{k,m} &= \begin{cases} 2 & \{i, j\} = \{k, m\} \text{ or } \{i, j\} = \{p, q\} \\ 0 & \text{otherwise} \end{cases}, \\
 \mu_{k,m}^{i,j} \cdot \mu_{r,s}^{p,q} &= \begin{cases} -2 & \{\{i, j\}, \{k, m\}\} = \{\{p, q\}, \{r, s\}\} \\ 2 & \{\{i, j\}, \{k, m\}\} \cap \{\{p, q\}, \{r, s\}\} = \emptyset \\ 0 & \text{otherwise} \end{cases}.
 \end{aligned}
 \tag{1.1}$$

TABLE 1. List of possible configurations

Class	Configuration of singular fibers	MWG
1.1	$I_{10} + I_2 + a II + b I_1$	$\mathbb{Z}^4$
1.2	$I_8 + I_4 + a II + b I_1$	$\mathbb{Z}^4$
1.3	$2 I_6 + a II + b I_1$	$\mathbb{Z}^4$
1.4	$IV^* + I_4 + a II + b I_1$	$\mathbb{Z}^5$
2.1	$II^* + 6 I_2 + 2 I_1$	$\{0\}$
2.2	$III^* + 7 I_2 + I_1$	$\mathbb{Z}/2\mathbb{Z}$
2.3	$III^* + I_0^* + 3 I_2 + 3 I_1$	$\{0\}$
2.4	$I_6^* + 4 I_2 + 4 I_1$	$\{0\}$
2.5	$I_4^* + 6 I_2 + 2 I_1$	$\mathbb{Z}/2\mathbb{Z}$
2.6	$I_4^* + I_0^* + 2 I_2 + 4 I_1$	$\{0\}$
2.7	$I_2^* + 8 I_2$	$(\mathbb{Z}/2\mathbb{Z})^2$
2.8	$I_2^* + I_0^* + 4 I_2$	$\mathbb{Z}/2\mathbb{Z}$
2.9	$2 I_2^* + 2 I_2 + 4 I_1$	$\{0\}$
2.10	$I_2^* + 2 I_0^* + 8 I_1$	$\{0\}$
2.11	$2 I_0^* + 6 I_2$	$(\mathbb{Z}/2\mathbb{Z})^2$
2.12	$3 I_0^* + 2 I_2 + 2 I_1$	$\mathbb{Z}/2\mathbb{Z}$

Moreover, for some rational plane curves  $C$ , we name the pull-back of  $C$  as in the following table. Note that all of them are  $(-2)$ -curves on  $X$ .

Divisor	$C$
$\eta_{(i_1 j_1)(i_2 j_2)(i_3 j_3)(i_4 j_4)(i_5 j_5)}$	the conic curve through $P_{i_1, j_1}, \dots, P_{i_5, j_5}$
$\xi_{(\overline{i_1 j_1})(i_2 j_2)(i_3 j_4)(i_5 j_5)(i_6 j_6)(i_7 j_7)}$	the cubic curve through $P_{i_1, j_1}, \dots, P_{i_7, j_7}$ with a double point at $P_{i_1, j_1}$
$\nu_{(\overline{i_1 j_1})(\overline{i_2 j_2})(\overline{i_3 j_3})(i_4 j_4)(i_5 j_5)(i_6 j_6)(i_7 j_7)(i_8 j_8)}$	the quartic curve through $P_{i_1, j_1}, \dots, P_{i_8, j_8}$ with a double point at $P_{i_1, j_1}, P_{i_2, j_2}, P_{i_3, j_3}$
$\gamma_{(\overline{i_1 j_1})(\overline{i_2 j_2})(\overline{i_3 j_3})(\overline{i_4 j_4})(i_5 j_5)(i_6 j_6)(i_7 j_7)(i_8 j_8)(i_9 j_9)}$	the quintic curve through $P_{i_1, j_1}, \dots, P_{i_9, j_9}$ with a triple point at $P_{i_1, j_1}$ and a double point at $P_{i_2, j_2}, P_{i_3, j_3}, P_{i_4, j_4}$

We may suppose that the lines  $L_i$  are defined by the following equations

$$(1.2) \quad \begin{aligned} L_1 : u &= 0, & L_2 : u &= 1, & L_3 : v &= 0, & L_4 : v &= 1 \\ L_5 : au + bv - 1 &= 0, & L_6 : cu + dv - 1 &= 0, \end{aligned}$$

where  $u, v$  are the affine parameters of projective plane. We consider the six lines

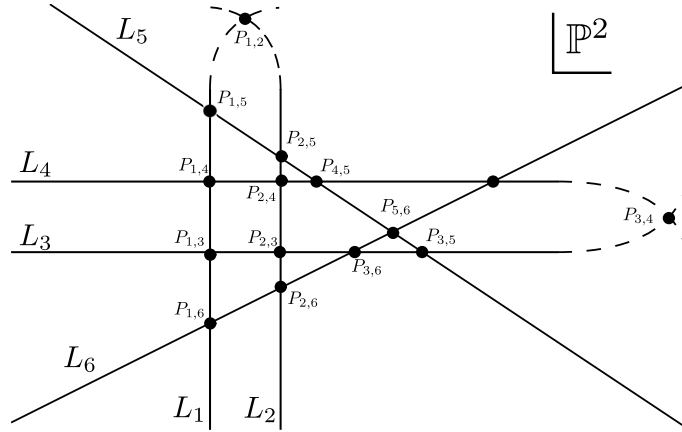


FIGURE 1. six lines on projective plane

in the *generic case*, that is, the rank of the Néron-Severi group  $NS(X)$  is 16.

Then the singular affine model of  $X$  is given by

$$(1.3) \quad w^2 = u(u-1)v(v-1)(au+bv-1)(cu+dv-1).$$

Under the above notation, we see that the divisors of typical functions are as follows.

$$\begin{aligned}
(u) &= 2\ell_1 + \ell_{1,3} + \ell_{1,4} + \ell_{1,5} + \ell_{1,6} - \left(\mu_{3,4}^{1,2} + \ell_{3,4}\right) \\
(u-1) &= 2\ell_2 + \ell_{2,3} + \ell_{2,4} + \ell_{2,5} + \ell_{2,6} - \left(\mu_{3,4}^{1,2} + \ell_{3,4}\right) \\
(v) &= 2\ell_3 + \ell_{1,3} + \ell_{2,3} + \ell_{3,5} + \ell_{3,6} - \left(\mu_{3,4}^{1,2} + \ell_{1,2}\right) \\
(v-1) &= 2\ell_4 + \ell_{1,4} + \ell_{2,4} + \ell_{4,5} + \ell_{4,6} - \left(\mu_{3,4}^{1,2} + \ell_{1,2}\right) \\
(1.4) \quad (au + bv - 1) &= 2\ell_5 + \ell_{1,5} + \ell_{2,5} + \ell_{3,5} + \ell_{4,5} + \ell_{5,6} - \left(\mu_{3,4}^{1,2} + \ell_{1,2} + \ell_{3,4}\right) \\
(cu + dv - 1) &= 2\ell_6 + \ell_{1,6} + \ell_{2,6} + \ell_{3,6} + \ell_{4,6} + \ell_{5,6} - \left(\mu_{3,4}^{1,2} + \ell_{1,2} + \ell_{3,4}\right) \\
(w) &= \sum_{i=1}^6 \ell_i + \sum_{1 \leq i < j \leq 6} \ell_{i,j} - 3 \left(\mu_{3,4}^{1,2} + \ell_{1,2} + \ell_{3,4}\right) \\
\left(\text{the equation of } M_{k,m}^{i,j}\right) &= \mu_{k,m}^{i,j} + \ell_{i,j} + \ell_{k,m} - \left(\mu_{3,4}^{1,2} + \ell_{1,2} + \ell_{3,4}\right).
\end{aligned}$$

For each class of Table 1, we compute a Weierstrass equation for one Jacobian fibration belonging to the class. Theoretically, constructing a Jacobian fibration on a  $K3$  surface is to find a divisor that has the same type as a singular fiber in the Kodaira's list (see [4]). In practice, however, we need to find two divisors, one for the fiber at  $t = 0$ , and the other for the fiber at  $t = \infty$ , to write down an actual elliptic parameter. Once an elliptic parameter is found, we would like to find a change of variables that converts to the defining equation to a Weierstrass form. In most cases, we encounter an equation of the form  $y^2 = (\text{quartic polynomial})$ . Then we can transform it to a Weierstrass form by using a standard algorithm (see for example [2] or [1]).

In our case, we use two methods to convert defining equation (1.3) to the form  $y^2 = (\text{quartic polynomial})$ . The first method is an elimination. Since an elliptic parameter  $t$  is a rational map, we can put  $t = f/g$  for some  $f, g \in k[u, v, w]$ . Thus, we can eliminate one variable from (1.3) and the equation  $gt - f = 0$ . If such an equation can be converted to  $y^2 = (\text{quartic polynomial})$  by a simple coordinate change, we can get a Weierstrass equation. We will call this method *classical method* in this paper.

The other method is a *2-neighbor step*, which is designed by Noam Elkies. This is the technique to transform a Weierstrass equation of a Jacobian fibration to a Weierstrass equation of a distinct Jacobian fibration. Using this, we can get an unknown Weierstrass equation of a class from a known class. We describe a 2-neighbor step in § 3.

**1.4. Results.** We state our main theorem.

**Theorem 1.** *Let  $X$  be a  $K3$  surface over an algebraically closed field  $k$  with  $\text{char}(k) \neq 2, 3$  that is the minimal resolution of the double covering of  $\mathbb{P}^2$  ramified along generic six lines. Suppose that the rank of the Néron-Severi group  $NS(X)$  is 16. Under the singular affine model (1.3) of  $X$ , for each class in Table 1, an elliptic parameter and a Weierstrass equation of a Jacobian fibration belonging to the class is given by Table 2, ..., Table 17.*

In the case of using a classical method, we give an elliptic parameter  $t$  in terms of  $u, v, w$ , a Weierstrass equation and a picture of the configuration of singular fibers. Moreover, for the class 2.xx, we also give the correspondence between the divisors and the torsion sections.

In the case of using a 2-neighbor step from another class (in fact, we only use a 2-neighbor step from the classes 2.7 or 2.5), we give an elliptic parameter  $s$  in terms of  $t, x, y$  used in the equation of the source class, a picture that shows the way to construct of the divisor corresponding to the fiber at  $s = \infty$  and a picture of the configuration of singular fibers. In this case, however, we omit Weierstrass equations, since they are all too long to print in this section. We give them in § 6.

We explain the detail of some computation. In § 2, we will give an elliptic parameter and a Weierstrass equation of a Jacobian fibration of the class 2.7 by a classical method. In § 3, we explain a 2-neighbor step. In § 4, we will give an elliptic parameter and a Weierstrass equation of the class 2.10 by a 2-neighbor step from the class 2.7. In § 5, we use a 2-neighbor step from the class 2.5 for the computation of the class 2.4.

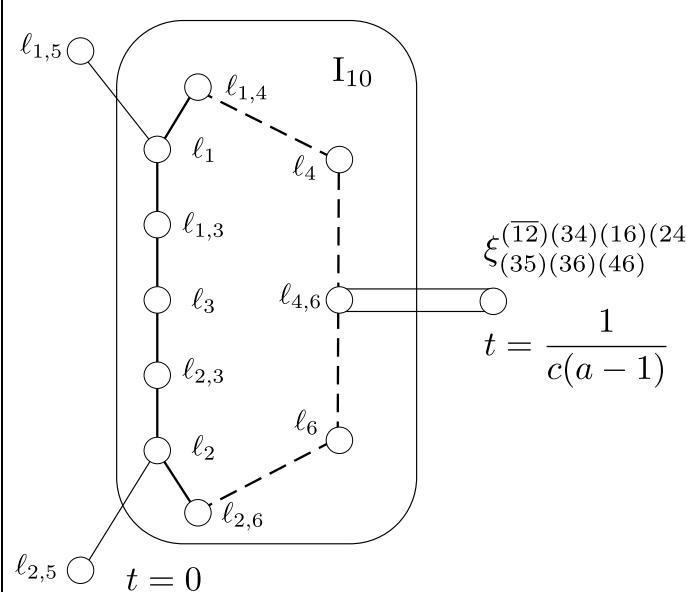
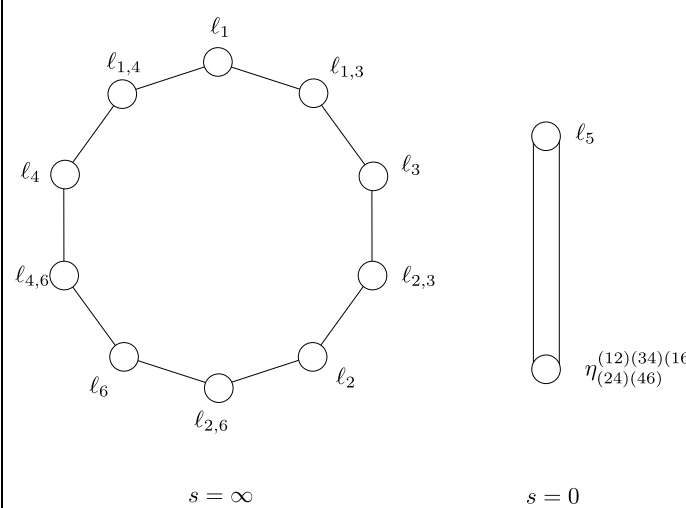
Class 1.1	
Method	2-neighbor step from the class 2.5
Elliptic parameter	$s = \frac{y}{tx(act - t - 1)}$
 <p style="text-align: right;"> <math>\xi_{(35)(36)(46)}^{(\overline{12})(34)(16)(24)}</math>  <math>t = \frac{1}{c(a-1)}</math> </p>	
 <p style="text-align: center;"><math>s = \infty</math></p> <p style="text-align: right;"><math>s = 0</math></p>	

TABLE 2. Class 1.1

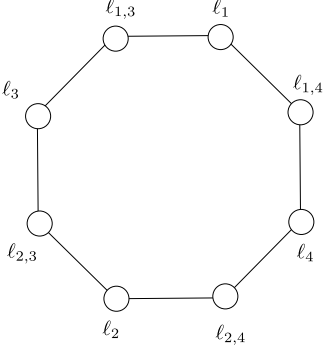
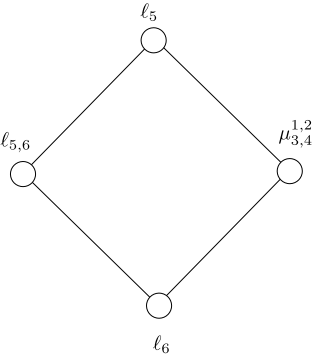
Class 1.2	
Method	Classical
Elliptic parameter	$t = \frac{w}{(au + bv - 1)(cu + dv - 1)}$
<div style="display: flex; justify-content: space-around; align-items: center;"> <div style="text-align: center;">  <p><math>t = 0</math></p> </div> <div style="text-align: center;">  <p><math>t = \infty</math></p> </div> </div>	
$ \begin{aligned} & y^2 + (2(b-d)(ad-bc)t^2 - 4 + 2b + 2d)xy \\ & + 4(b-d)^4 t^2 ((2a + 2ac + 2c - ad - bc)t^2 + 1)y \\ & = x^3 - 2((b-d)(b^2 + 2abd - 2bcd - 2b - 2ab + 2cd - d^2 + 2d)t^2 \\ & + 2(d-1)(b-1))x^2 - 4t^4(b-d)^6(4act^2 + 1)x \\ & + 8t^4(b-d)^6(4act^2 + 1)((b-d)(b^2 + 2abd - 2bcd - 2b - 2ab \\ & + 2cd - d^2 + 2d)t^2 + 2(d-1)(b-1)) \end{aligned} $	

TABLE 3. Class 1.2



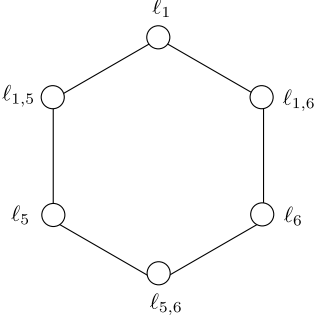
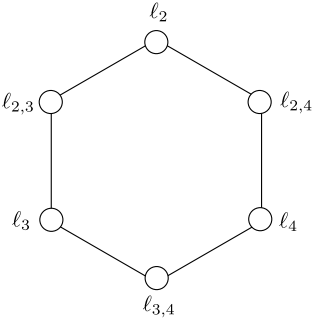
Class 1.3	
Method	Classical
Elliptic parameter	$t = \frac{w}{v(u-1)(v-1)}$
<div style="display: flex; justify-content: space-around; align-items: center;"> <div style="text-align: center;">  <p><math>t = 0</math></p> </div> <div style="text-align: center;">  <p><math>t = \infty</math></p> </div> </div>	
$t^2 v - u + (d + b - t^2) uv + (t^2 - bd) uv^2 - (bc + ad) u^2 v - t^2 v^2 - acu^3 + (a + c) u^2 = 0$ <p>This converts to a Weierstrass form (see [2] or [1]).</p>	

TABLE 4. Class 1.3

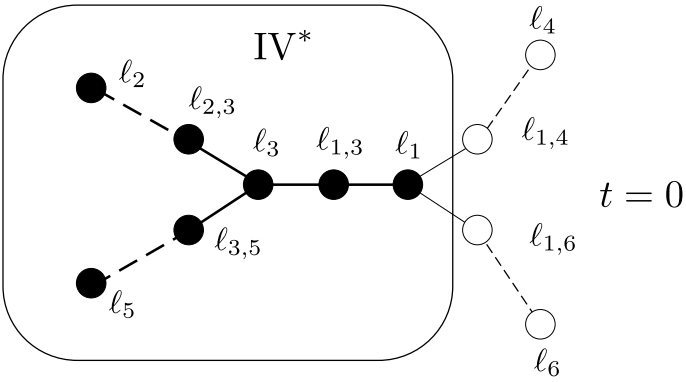
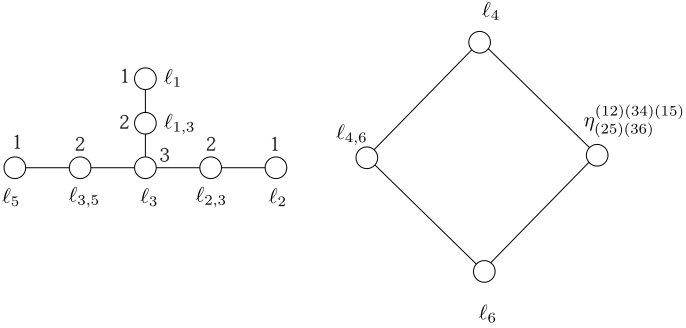
Class 1.4	
Method	2-neighbor step from the class 2.7
Elliptic parameter	$s = \frac{y}{t^2(x - at(bt - dt - 1)(bt + ct - t - 1))}$
	
 <div style="display: flex; justify-content: space-around; margin-top: 10px;"> <div style="text-align: center;"> <math>s = \infty</math> </div> <div style="text-align: center;"> <math>s = 0</math> </div> </div>	

TABLE 5. Class 1.4

Class 2.1	
Method	2-neighbor step from the class 2.5
Elliptic parameter	$s = \frac{x - (b-1)(ad - bc + b - d)t^3}{t^4}$
<p>● : identity component</p> <p> </p> <p> <math>\gamma_{(15)(16)(35)(46)(56)}^{(\overline{12})(\overline{34})(\overline{24})(\overline{36})}</math> <math>\eta_{(35)(36)}^{(12)(16)(24)}</math> <math>s = 0</math> </p> <p> <math>\gamma_{(15)(16)(34)(46)(56)}^{(\overline{12})(\overline{24})(\overline{35})(\overline{36})}</math> <math>\eta_{(24)(36)}^{(12)(34)(16)}</math> <math>s = a(c+d-1)(b-d)(a+b-bc-1)</math> </p> <p> <math>\nu_{(16)(35)(56)}^{(\overline{12})(\overline{24})(\overline{36})(15)}</math> <math>\xi_{(35)(36)(46)}^{(\overline{12})(\overline{34})(16)(24)}</math> <math>s = c(b-d)(1-a)(ad-bc+b+c-1)</math> </p> <p> <math>\nu_{(16)(35)(46)}^{(\overline{12})(\overline{24})(\overline{36})(15)}</math> <math>\xi_{(35)(36)(56)}^{(\overline{12})(\overline{34})(16)(24)}</math> <math>s = (a-1)(b-1)(ad-bc)(1-c-d)</math> </p> <p> <math>\ell_{4,5}</math> <math>N_7</math> <math>s = a(bc-c-d+1)(bc-ad-b+d)</math> </p> <p> <math>\mu_{3,6}^{1,2}</math> <math>N_6</math> <math>s = c(b-1)(1-a-d)(ad-bc+b-d)</math> </p> <p><math>N_i</math> : the pull-back of a rational plane curve of degree <math>i</math></p>	

TABLE 6. Class 2.1

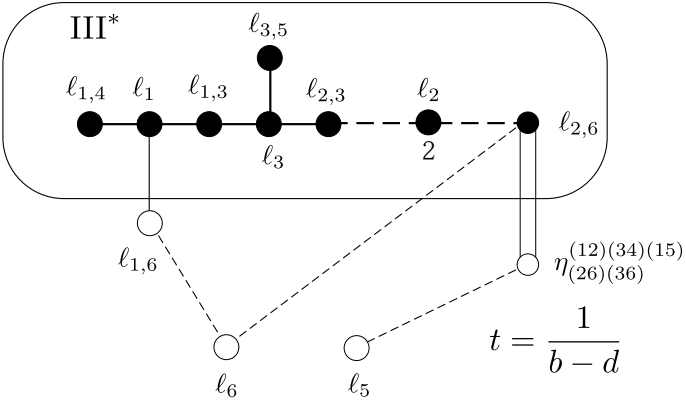
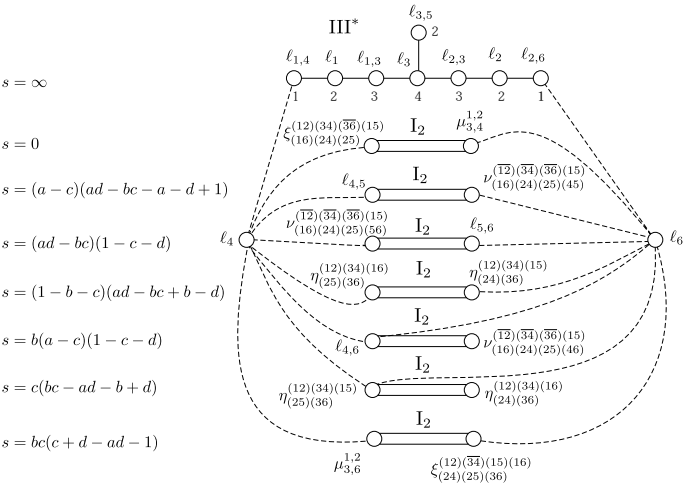
Class 2.2	
Method	2-neighbor step from the class 2.7
Elliptic parameter	$s = \frac{x + (1-a)t + A_2 t^2 - A_3 t^3}{t^3(bt - dt - 1)},$ $A_2 = 2ab - bc + ac - a - b - c + 1$ $A_3 = (b-d)(ab - bc + ac + ad - a - c - d + 1)$
<p style="text-align: center;"><math>t = 0</math></p> 	
<p style="text-align: center;"><math>s = \infty</math></p> <p style="text-align: center;"><math>s = 0</math></p> <p style="text-align: center;"><math>s = (a-c)(ad - bc - a - d + 1)</math></p> <p style="text-align: center;"><math>s = (ad - bc)(1 - c - d)</math></p> <p style="text-align: center;"><math>s = (1 - b - c)(ad - bc + b - d)</math></p> <p style="text-align: center;"><math>s = b(a-c)(1 - c - d)</math></p> <p style="text-align: center;"><math>s = c(bc - ad - b + d)</math></p> <p style="text-align: center;"><math>s = bc(c + d - ad - 1)</math></p> 	

TABLE 7. Class 2.2

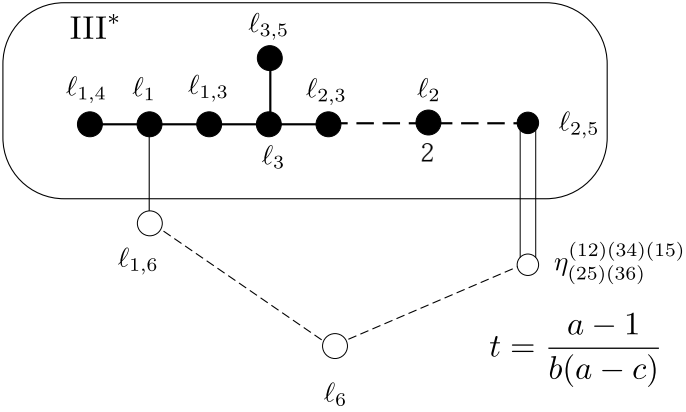
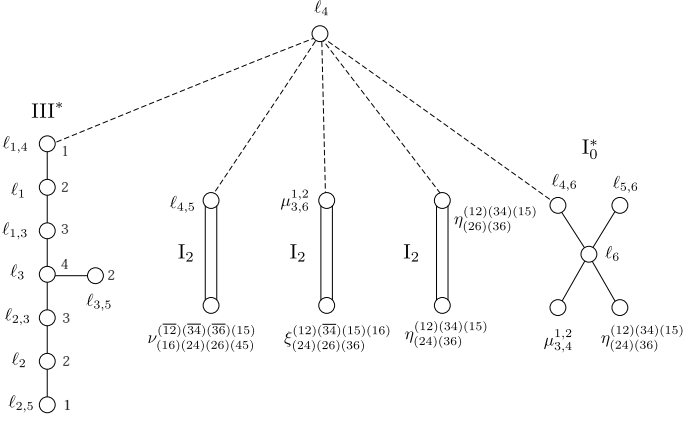
Class 2.3	
Method	2-neighbor step form the class 2.7
Elliptic parameter	$s = \frac{x - (a-1)t + A_2 t^2 - b(c+b-1)(a-c)t^3}{t^3(b(a-c)t - a + 1)},$ $A_2 = 2ab - bc + ac - a - b - c + 1$
<p style="text-align: center;"><math>t = 0</math></p>  <p style="text-align: center;"><math>t = \frac{a-1}{b(a-c)}</math></p>	
 <p style="text-align: center;"> <math>s = \infty \quad s = ad - bc - a + c \quad s = cd(1-b) \quad s = (b-d)(1-c-d) \quad s = 0</math> </p>	

TABLE 8. Class 2.3

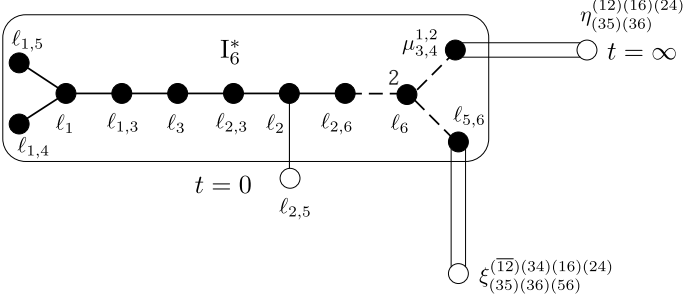
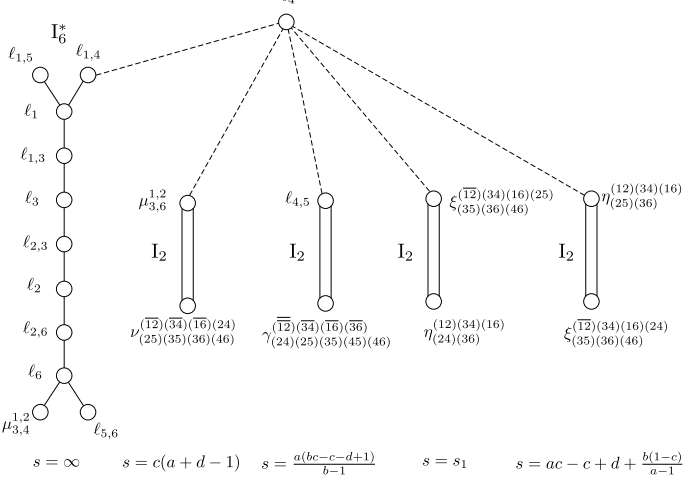
Class 2.4	
Method	2-neighbor step from the class 2.5
Elliptic parameter	$s = \frac{(ad-bc)(x-t) + (a-1)(ad-bc)(c+d-1)t^2}{t^2((a-1)(ad-bc)(c+d-1)t - ad + bc - b + d)}$
 $t = \frac{ad-bc+b-d}{(a-1)(ad-bc)(c+d-1)}$	
 $s = \infty \quad s = c(a+d-1) \quad s = \frac{a(bc-c-d+1)}{b-1} \quad s = s_1 \quad s = ac - c + d + \frac{b(1-c)}{a-1}$	
$s_1 = \frac{(c^2 + 2cd - 2c - d + 1)a + (d-1)(bc - b + d)}{c + d - 1}$	

TABLE 9. Class 2.4

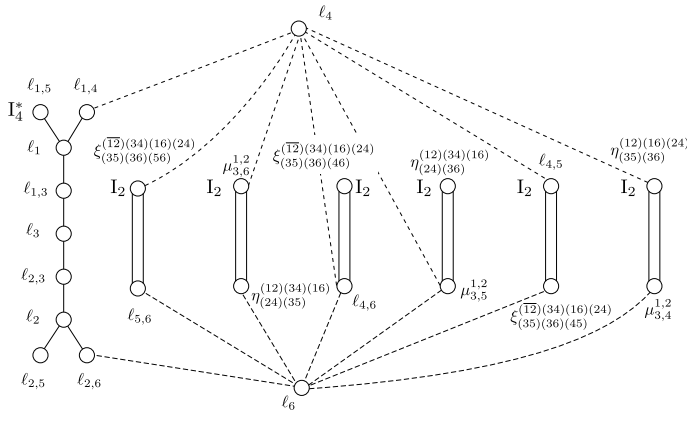
Class 2.5	
Method	Classical
Elliptic parameter	$t = \frac{-uv(u-1)}{(a+c+d-ac-1)uv+acu^2-(a+c)u-dv+1}$
 <p style="text-align: center;"> <math>t = 0 \quad t = t_1 \quad t = \frac{1}{c(a+d-1)} \quad t = \frac{1}{c(a-1)} \quad t = \frac{1}{a(c+d-1)} \quad t = t_2 \quad t = \infty</math> <math display="block">t_1 = \frac{ad-bc+b-d}{(a-1)(ad-bc)(c+d-1)}, \quad t_2 = \frac{b-1}{a(bc-c-d+1)}</math> </p>	
$y^2 = x^3 + t \left( ac(a-1)(c+d-1)(ac+bc-a-c-d+1)t^3 \right. \\ + (4ac^2 - c^2 + a - 5ac - cd - a^2 + 2acd + 2bcd - ad^2 - 3a^2c^2 \\ + bc^2 - 2a^2cd + c + 3abc - abcd + a^2d - 2bc + 4a^2c - 2bc^2a)t^2 \\ + (ad + bc + 3ac - 2a - 2b - 2c + d + 1)t - 1 \Big) x^2 \\ - t^4 \left( c(a+d-1)t - 1 \right) \left( a(bc-c-d+1)t - b + 1 \right) \left( (a-1) \right. \\ \left. (ad-bc)(c+d-1)t - ad + bc - b + d \right) x$	
Zero section	$\ell_6$
2-torsion section	$\ell_4 : (x, y) = (0, 0)$

TABLE 10. Class 2.5

TABLE 11. Class 2.6



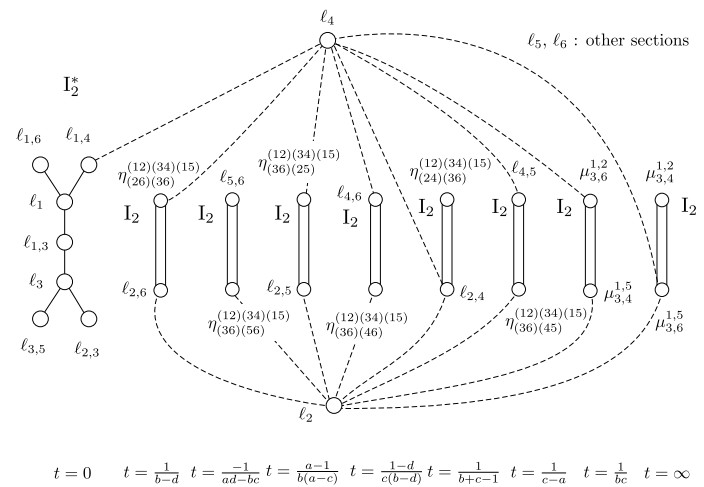
Class 2.7	
Method	Classical
Elliptic parameter	$\frac{uv}{cu + bv - 1}$
$y^2 = (x - t(bt + ct - t - 1)(abt - bct - a + 1))$ $(x - at(bt - dt - 1)(bt + ct - t - 1))$ $(x - t(1 - ct)(bt - dt - 1)(abt - bct - a + 1)),$	
 <p style="text-align: center;"> <math>t = 0 \quad t = \frac{1}{b-d} \quad t = \frac{-1}{ad-bc} \quad t = \frac{a-1}{b(a-c)} \quad t = \frac{1-d}{c(b-d)} \quad t = \frac{1}{b+c-1} \quad t = \frac{1}{c-a} \quad t = \frac{1}{bc} \quad t = \infty</math> </p>	
Zero section	$\ell_2$
2-torsion section	$\ell_4 : (x, y) = (t(1 - ct)(abt - bct - a + 1)(bt - dt - 1), 0)$ $\ell_5 : (x, y) = (at(bt - dt - 1)(bt + ct - t - 1), 0)$ $\ell_6 : (x, y) = (t(abt - bct - a + 1)(bt + ct - t - 1), 0)$

TABLE 12. Class 2.7

Class 2.8	
Method	Classical
Elliptic parameter	$t = \frac{u(v-1)}{cu+dv-1}$
$y^2 = x^3 + t((ad+bc+acd-bc^2-2cd)t^2$ $+ (ad-2bc-2a+b+c-2d+1)t-b+1)x^2$ $+ t^3(cdt+d-1)(at-ct-1)(adt-bct+bt-dt-a-b+1)x$	
<p style="text-align: center;"> <math>t = 0 \quad t = \frac{1}{a-c} \quad t = \frac{1}{1-c} \quad t = \frac{1-d}{cd} \quad t = \frac{a+b-1}{ad-bc+b-d} \quad t = \infty</math> </p>	
Zero section	$\ell_2$
2-torsion section	$\ell_3 : (x, y) = (0, 0)$

TABLE 13. Class 2.8

Class 2.9	
Method	Classical
Elliptic parameter	$t = \frac{u(au + bv - 1)}{(u - 1)(v - 1)(au - 1)}$
<p style="text-align: center;"> <math>t = 0</math> <math>t = \frac{1}{c-1}</math> <math>t = \frac{b-1}{a+b-1}</math> <math>t = \infty</math> </p>	
$y^2 = x^3 + t(t+1)((a-1)(c+d-1)t^3 - (2ac - 2ad + 2bc - b + 3d - 2)t^2 + (ad - 2bc + a + 2b + c - 3d + 1)t^2 + b - d)x^2 - t^3(t+1)^2(ct - t - 1)((a+b-1)t + b - 1)((2ac + ad - a - c)t^2 - (ac - 2ad + bc + a + c)t + ad - bc)x + act^6(t+1)^3(ct - t - 1)^2(at + bt - t + b - 1)^2$	
Zero section	$\ell_3$

TABLE 14. Class 2.9

TABLE 15. Class 2.10

Class 2.11	
Method	Classical
Elliptic parameter	$t = u$
<p> <math>\ell_{1,3}</math> <math>\ell_{2,3}</math>  <math>\ell_{1,4}</math> <math>\ell_1</math> <math>\ell_{1,6}</math> <math>\ell_{2,4}</math> <math>\ell_2</math> <math>\ell_{2,6}</math>  <math>t = 0</math> <math>t = 1</math>  <math>\ell_{1,5}</math> <math>\ell_{2,5}</math>  <math>\ell_{3,5}</math> <math>\ell_{3,6}</math> <math>\ell_{4,5}</math> <math>\ell_{4,6}</math> <math>\ell_{5,6}</math> <math>\ell_{3,4}</math>  <math>\mu_{3,5}^{1,2}</math> <math>\mu_{3,6}^{1,2}</math> <math>\mu_{4,5}^{1,2}</math> <math>\mu_{4,6}^{1,2}</math> <math>\mu_{5,6}^{1,2}</math> <math>\mu_{3,4}^{1,2}</math>  <math>t = \frac{1}{a}</math> <math>t = \frac{1}{c}</math> <math>t = \frac{1-b}{a}</math> <math>t = \frac{1-d}{c}</math> <math>t = \frac{b-d}{ad-bc}</math> <math>t = \infty</math> </p>	
$y^2 = (x - dt(t-1)(at-1))$ $(x - bt(t-1)(ct-1))$ $(x - t(t-1)(at-1)(ct-1))$	
Zero section	$\ell_3$
2-torsion section	$\ell_4 : (x, y) = (t(1-t)(ct-1)(at-1), 0)$ $\ell_5 : (x, y) = (bt(t-1)(ct-1), 0)$ $\ell_6 : (x, y) = (dt(t-1)(at-1), 0)$

TABLE 16. Class 2.11

Class 2.12	
Method	Classical
Elliptic parameter	$t = \frac{u(bv + a - 1)}{au + bv - 1}$
<p style="text-align: center;"> <math>t = 0 \qquad t = 1 \qquad t = \infty \qquad t = \frac{a-1}{a-c} \qquad t = \frac{(d-1)(a+b-1)}{ad-bc-a+c}</math> </p>	
$y^2 = x^3 + 9t(t-1)((ad-bc-2a+2c)t - ad + 2a + b + d - 2)x^2 - 81t^2(t-1)^2(at-ct-a+1)((ad-bc-a+c)t - (d-1)(a+b-1))x$	
Zero section	$\ell_3$
2-torsion section	$\ell_4 : (x, y) = (0, 0)$

TABLE 17. Class 2.12

## 2. CLASS 2.7

An elliptic parameter for the class 2.7 is given by

$$(2.1) \quad t = \frac{uv}{cu + bv - 1}.$$

It is easy to verify that the divisor of  $t$  is given by

$$(2.2) \quad (t) = \ell_{1,6} + \ell_{1,4} + 2(\ell_1 + \ell_{1,3} + \ell_3) + \ell_{2,3} + \ell_{3,5} - (\mu_{3,6}^{1,5} + \mu_{3,4}^{1,2}).$$

Then the fiber at  $t = 0$  is of type  $I_2^*$  and  $t = \infty$  fiber is of type  $I_2$ .

Eliminating the variable  $v$  from (1.3) and (2.1), and making a simple coordinate change, we obtain an equation of the form  $y^2 = (\text{quartic polynomial})$ . Choosing  $\ell_2$  as the zero section of the group structure, we have the Weierstrass equation for the Jacobian fibration

$$(2.3) \quad \begin{aligned} y^2 = & (x - t(bt + ct - t - 1)(abt - bct - a + 1)) \\ & (x - at(bt - dt - 1)(bt + ct - t - 1)) \\ & (x - t(1 - ct)(bt - dt - 1)(abt - bct - a + 1)), \end{aligned}$$

where the change of variables is given by

$$(2.4) \quad \begin{aligned} x = & \frac{t(bt - dt - 1)(bt + ct - t - 1)(abt - bct - a + 1)}{u - 1} \\ y = & \frac{t(bt - dt - 1)(bt + ct - t - 1)(abt - bct - a + 1)(u - bt)^2 w}{u(cu - 1)(u - 1)^2}. \end{aligned}$$

The remaining of divisors  $\ell_4, \ell_5$  and  $\ell_6$  are 2-torsion sections. The correspondence between the divisors and the sections are as follows.

$$(2.5) \quad \begin{aligned} \ell_4 & \leftrightarrow (x, y) = (t(1 - ct)(abt - bct - a + 1)(bt - dt - 1), 0) \\ \ell_5 & \leftrightarrow (x, y) = (at(bt - dt - 1)(bt + ct - t - 1), 0) \\ \ell_6 & \leftrightarrow (x, y) = (t(abt - bct - a + 1)(bt + ct - t - 1), 0) \end{aligned}$$

We can check types and positions of other singular fibers by Tate algorithm [10]. For example, the fiber at  $t = \frac{1}{bc}$  is a singular fiber of type  $I_2$ . Since we obtain

$$(2.6) \quad \left(t - \frac{1}{bc}\right) = \left(\frac{(cu - 1)(bv - 1)}{(cu + bc - 1)cb}\right) = \mu_{3,6}^{1,2} + \mu_{3,4}^{1,5} - (\mu_{3,4}^{1,2} + \mu_{3,6}^{1,5})$$

by (2.1) and (1.4), we see that the fiber at  $t = \frac{1}{bc}$  consists of  $\mu_{3,6}^{1,2}$  and  $\mu_{3,4}^{1,5}$ .

Similarly, we can determine the other singular fibers. In the following Figure 2, we show the complete configuration of singular fibers for the class 2.7.

## 3. 2-NEIGHBOR STEP

In this section, we explain “2-neighbor step”. The following description is based on [5].

Let  $X$  be a  $K3$  surface over a field  $k$  with an Jacobian fibration over  $\mathbb{P}^1$  with a zero section  $O$ , which defines an elliptic curve  $E$  over  $k(t)$ . Let  $P, Q$  be other sections. Let  $F$  be the class of a fiber. We call  $F$  an *elliptic divisor* of this fibration. Then for an effective divisor  $F' = mO + nP + kQ + G$ , where  $G$  is an effective vertical divisor, we would like to compute the global sections of  $\mathcal{O}_X(F')$ . Denote

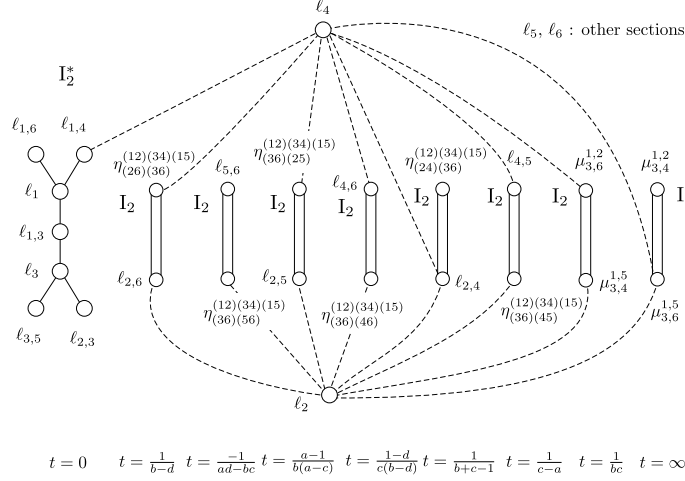


FIGURE 2. configuration of singular fibers for the class 2.7

the space of the global sections of  $\mathcal{O}_X(F')$  by  $L_X(F')$ . Let  $\{s_1, \dots, s_r\}$  be a basis of  $L_E(mO + nP + kQ)$ . Then for any  $f \in L_X(F')$ , there exist  $b_i(t) \in k(t)$  such that

$$(3.1) \quad f = b_1(t)s_1 + \dots + b_r(t)s_r.$$

We suppose that there exists an elliptic divisor  $F'$  on  $X$  with  $F' \cdot F = 2$ . Then decomposing  $F'$  into horizontal and vertical components  $F' = F'_h + F'_v$ , we see that  $F'_h \cdot F + F'_v \cdot F = F' \cdot F = 2$ . Since  $F'_v \cdot F = 0$ , we have  $F'_h \cdot F = 2$ . Therefore the possibilities are  $F'_h = 2P$  or  $F'_h = P + Q$  for sections  $P, Q$ . By a translation, we can take the first one to be  $2O$ , and the second to be  $O + T$  with a 2-torsion section  $T$  or  $O + P$  with a non 2-torsion section  $P$  depending on whether the class of the section  $[P - Q]$  is 2-torsion or not.

We suppose that a Weierstrass equation of  $E/k(t)$  is given by

$$(3.2) \quad y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$

with  $a_i(t) \in k[t]$  of degree at most  $2i$ .

First, we consider the case  $D = 2O$ . Then 1 and  $x$  form a basis for  $L_E(D)$ . Therefore, for a new elliptic divisor  $F' = 2O + G$  with  $G$  effective and vertical, we obtain two elements 1 and  $A(t) + B(t)x$  of  $L_X(F')$  for some fixed  $A(t), B(t) \in k(t)$ . The ratio of the two global sections gives the new elliptic parameter  $s = A(t) + B(t)x$ . Therefore we set

$$(3.3) \quad x = \frac{s - A(t)}{B(t)}$$

and substitute this into the Weierstrass equation, to get the form of

$$(3.4) \quad y^2 = g(t, s).$$

Since the generic fiber of the fibration over  $\mathbb{P}_s^1$  is a curve of genus 1, after absorbing square factors into  $y^2$ ,  $g$  must be a polynomial of degree 3 or 4 in  $t$ .

Next, we consider the case  $D = O + P$  where  $P = (x_0, y_0)$  is not a 2-torsion section. Then 1 and  $\frac{y+y_0}{x-x_0}$  form a basis of  $L_E(D)$ . Therefore we obtain a new elliptic



parameter

$$(3.5) \quad s = A(t) + B(t) \frac{y + y_0}{x - x_0}.$$

Solving (3.5), we get

$$(3.6) \quad y = \frac{(s - A(t))(x - x_0)}{B(t)} - y_0.$$

Substituting into the Weierstrass equation (3.2) we get

$$(3.7) \quad \left( \frac{(s - A(t))(x - x_0)}{B(t)} - y_0 \right)^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6.$$

Since  $x_0$  and  $y_0$  satisfy the Weierstrass equation (3.2), the difference of the left and right hand sides of this equation can be divided by  $(x - x_0)$ . Therefore we get an equation  $g(x, t, s) = 0$  that is quadratic in  $x$ . By completing the square, we obtain an equation of the form

$$(3.8) \quad x^2 = h(t, s)$$

and, after absorbing square factors into  $x$ , we have that  $h$  is cubic or quartic in  $t$ .

Finally, we consider the case  $D = O + T$  where  $T$  is a 2-torsion section. In this case, we may assume that  $T = (0, 0)$  and  $a_6 = 0$  by a translation. Then 1 and  $\frac{y}{x}$  form a basis of  $L_E(D)$ . Setting a new elliptic parameter  $s = A(t) + B(t)\frac{y}{x}$ , we obtain

$$(3.9) \quad y = \frac{(s - A(t))x}{B(t)}.$$

Substituting this into the Weierstrass equation (3.2), we have

$$(3.10) \quad \left( \frac{(s - A(t))x}{B(t)} \right)^2 = x^3 + a_2(t)x^2 + a_4(t)x.$$

Dividing both sides by  $x$ , we obtain a quadratic equation, and we can proceed as in the previous case.

#### 4. CLASS 2.10

To obtain the Weierstrass equation for the class 2.10, we use a 2-neighbor step from the class 2.7. We compute explicitly the elements of  $L_X(F')$  where

$$(4.1) \quad F' = 2\ell_2 + \ell_{2,3} + \ell_{2,4} + \mu_{3,6}^{1,5} + \eta_{(36)(56)}^{(12)(34)(15)}$$

is the class of the fiber of type  $I_0^*$  we are considering. The linear space  $L(F')$  is 2-dimensional, and the ratio of two linearly independent elements will be an elliptic parameter for  $X$ . Thus, we may find a non-constant rational function on  $X$  belonging to  $L(F')$ , for which 1 is an element of  $L(F')$ . Then it will be an elliptic parameter of fibration 2.10. Let  $s \in L(F')$  be a non-constant. Notice that  $s$  has a pole of order 2 along  $\ell_2$ , which is the zero section of fibration 2.7. Also, it has a simple pole along  $\ell_{2,4}$ , the identity component of the fiber at  $t = \frac{1}{b+c-1}$ , a simple pole along  $\mu_{3,6}^{1,5}$ , the identity component of the fiber at  $t = \infty$ , and a simple pole along  $\eta_{(36)(56)}^{(12)(34)(15)}$ , the identity component of the fiber at  $t = \frac{-1}{ad-bc}$ . Therefore we can put

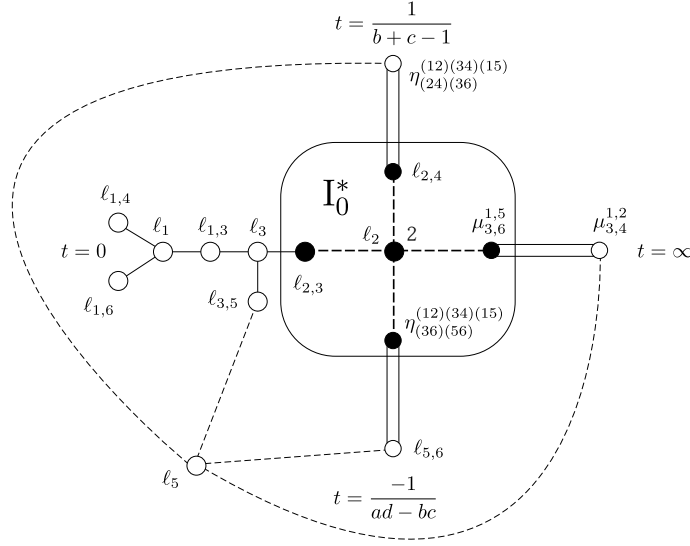


FIGURE 3. 2-neighbor step from class 2.7 to class 2.10

$$(4.2) \quad s = \frac{x + A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4}{t(t + \frac{1}{ad-bc})(t - \frac{1}{b+c-1})}.$$

We may subtract a term  $A_3 t \left(t + \frac{1}{ad-bc}\right) \left(t - \frac{1}{b+c-1}\right)$  from the numerator, since 1 is an element of  $L(F')$ . Thus, assume  $A_3 = 0$ . To obtain other coefficients the  $A_i$ , we look at the order of vanishing along the non-identity components of fibers at  $t = 0, \infty, \frac{-1}{ad-bc}, \frac{1}{b+c-1}$ . For example, we look at the fiber at  $t = 0$ . The rational function  $s$  does not have any pole along  $\ell_{3,5}$ , which intersects with the section  $\ell_5$  of the fibration for the class 2.7 at  $t = 0$ . Hence  $s$  has no pole at  $t = 0$  and  $x = -at(bt - dt + 1)(bt + ct - t + 1)$ , which corresponds to the section  $\ell_5$ , and that gives us  $A_0$ . Similarly, other fibers give remaining coefficients. After some calculation, we get a new elliptic parameter

$$(4.3) \quad s = \frac{(ad-bc)x - a(ad-bc+b-d)t + a(c+b-1)(ad-bc+b-d)t^2}{t((ad-bc)t+1)((b+c-1)t-1)}.$$

Solving for  $x$  in terms of  $s$  and substituting into right hand side of (2.3), and dividing out some square factors, which can absorb into  $y^2$ , we obtained an equation for fibration 2.10 in the form  $y^2 = f(t)$ , where  $f$  is a quartic in  $t$  with coefficients in  $k(a, b, c, d, s)$ . In fact, since the quartic  $f$  factors into  $t$  times cubic factor in  $t$ , the new equation converts to a Weierstrass form by a standard algorithm. Finally, we

get a Weierstrass equation of the fibration for the class 2.7.

$$\begin{aligned}
 Y^2 = & X^3 + (ad - bc)(s + ad - ab)(s - ab - bc)((b + c - ad + bc - 1)s \\
 & + b^2c^2 + b^2c - 2abc - a^2d^2 - bcd + a^2d - ad - ab^2 + ad^2 \\
 & + bc^2 + ab - ab^2c + a^2bd) X^2 \\
 & - (s - ab + ad)^2(s - ab + bc)^3(ad - bc)^3(s^3 - 3b(a - c)s^2 \\
 & + 3b^2(a - c)^2s - b^2(a - c)^3) X \\
 & + bc(a - c)(b - d)(ad - bc)^4(s - ab + ad)^3(s - ab + bc)^3.
 \end{aligned}
 \tag{4.4}$$

The new variables  $X, Y$  are related to the ones from fibration 2.7 by the following equation.

$$\begin{aligned}
 X &= \frac{(ad - bc)(ab - ad - s)(ab - bc - s)^2}{t} \\
 Y &= \frac{(ad - bc)^3(ab - ad - s)(ab - bc - s)^2y}{t^2}.
 \end{aligned}
 \tag{4.5}$$

The elliptic parameter  $s$  for the class 2.10 is also given in terms of  $u, v$  by

$$s = \frac{g}{(u - 1)(audv - bvcu + cu + bv - 1)(cu + bv - 1)},
 \tag{4.6}$$

where  $g$  is given by

$$\begin{aligned}
 g = & b(a - c)(b - d)(ad - bc)u^2v^2 + ac^2(ad - bc + b - d)u^3 \\
 & + c(2ab^2 - b^2c^2 - abd - b^2c + bcd - ad^2 + a^2d^2)u^2v \\
 & - b(b^2c^2 + ad^2 - a^2d^2 + b^2c + a^2bd - bcd - ab^2 - ab^2c)uv^2 \\
 & + c(bc^2 + abc + 2ad - 2a^2d - 2ab)u^2 - b^3(a - c)v^2 \\
 & + (ad^2 - 2ab^2c + b^2c - 2ab^2 + abd + 3b^2c^2 - a^2d^2 - bcd)uv \\
 & + (ab - ad - 2bc^2 + abc + a^2d)u + 2b^2(a - c)v - b(a - c),
 \end{aligned}
 \tag{4.7}$$

which defines the rational plane curve through  $P_{1,2}, P_{3,4}, P_{1,5}, P_{2,5}, P_{2,6}, P_{3,6}$  and  $P_{5,6}$  with double points at  $P_{1,2}, P_{1,5}$  and  $P_{3,6}$ .

The reducible fibers are as follows:

Position	Reducible fiber	Type
$s = \infty$	$2\ell_2 + \ell_{2,4} + \mu_{3,6}^{1,5} + \eta_{(36)(56)}^{(12)(34)(15)}$	$\Gamma_0^*$
$s = b(c - a)$	$2\ell_5 + \ell_{4,5} + \mu_{3,6}^{1,2} + \eta_{(26)(36)}^{(12)(34)(15)}$	$\Gamma_0^*$
$s = a(b - d)$	$\ell_{1,4} + \ell_{1,3} + 2(\ell_1 + \ell_{1,3} + \ell_6) + \ell_{4,6} + \eta_{(25)(36)}^{(12)(34)(15)}$	$\Gamma_2^*$

Then  $\ell_3$  is the zero section and  $\ell_4$  is a tri-section.

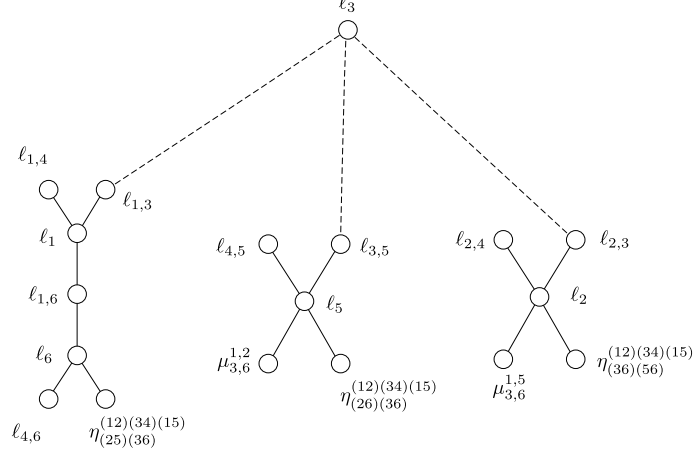
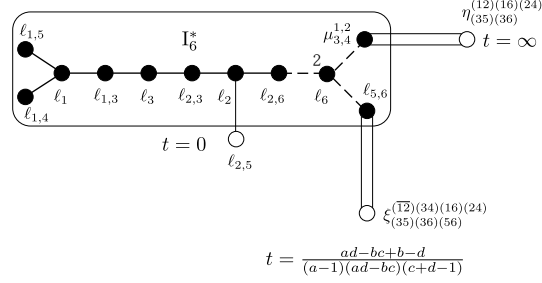


FIGURE 4. Configuration of singular fibers for the class 2.10

## 5. CLASS 2.4

The Weierstrass equation for the class 2.4 is obtained from class 2.5 by using a 2-neighbor step as the following.



A new elliptic divisor is

(5.1)

$$F' = \ell_{1,5} + \ell_{1,4} + 2(\ell_1 + \ell_{1,3} + \ell_3 + \ell_{2,3} + \ell_2 + \ell_{2,6} + \ell_6) + \mu_{3,4}^{1,2} + \xi_{(35)(36)(56)}^{(\overline{12})(34)(16)(24)}.$$

Then, we can put a new elliptic parameter  $s$  to

(5.2)

$$s = \frac{x + A_0 + A_1 t + A_2 t^2}{t^2 \left( t - \frac{ad-bc+b-d}{(a-1)(ad-bc)(c+d-1)} \right)}.$$

After some scaling for  $s$ , looking at the order of vanishing along the non-identity components of fibers at  $t = 0$  and  $t = \frac{ad-bc+b-d}{(a-1)(ad-bc)(c+d-1)}$ , we get

(5.3)

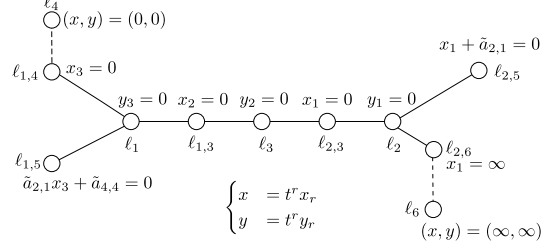
$$A_0 = 0, \quad A_2 = \frac{(1-a)(ad-bc)(c+d-1)}{ad+bc+b-d} A_1.$$

To determine the coefficient  $A_1$ , we need look at the order of vanishing along the non-identity component  $\ell_{2,5}$  of the fiber at  $t = 0$ . Now, we denote the equation for the class 2.5, for short, by

(5.4)

$$y^2 = x^3 + a_2 x^2 + a_4 x.$$

The equation for each component of the fiber at  $t = 0$  of type  $I_4^*$  is given by the following (see [9, IV §9] for detail)



where  $a_{i,r} = t^{-r} a_i$  and  $\tilde{a}_{i,r} = a_{i,r}(0)$ . In this case, we see that  $\tilde{a}_{2,1} = -1$ ,  $\tilde{a}_{4,4} = (b-1)(ad-bc+b-d)$ . Thus, substituting  $x = tx_1$  and  $x_1 = -1$ , we see that

$$(5.5) \quad s = \frac{1 + A_1 + A_2 t^2}{t((a-1)(ad-bc)(c+d-1)t - (ad-bc+b-d))}$$

on  $\ell_{2,5}$ . Since  $s$  has a pole along  $\ell_{2,5}$  unless  $A_1 = -1$ , we get  $A_1 = -1$ .

As a consequence, we have a new elliptic parameter

$$(5.6) \quad s = \frac{(ad-bc)(x-t) + (a-1)(ad-bc)(c+d-1)t^2}{t^2((a-1)(ad-bc)(c+d-1)t - ad+bc-b+d)}.$$

Therefore, we can compute a Weierstrass equation for the class 2.4 by using a 2-neighbor step from the class 2.5. However, we omit it, since it is too long to write down here. The configuration of the class 2.4 is the following.

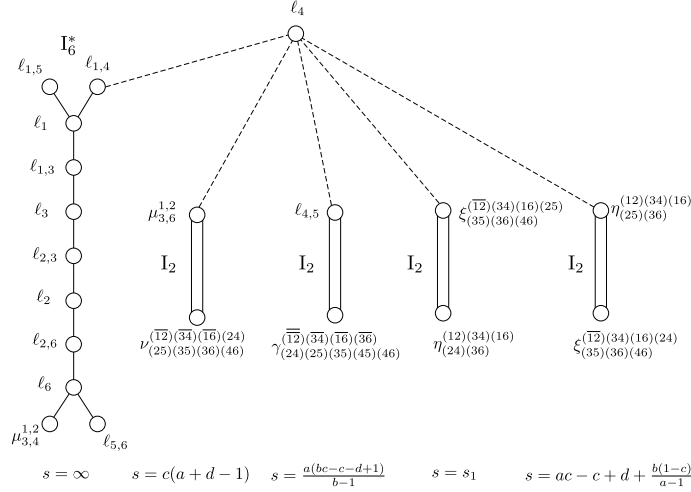


FIGURE 5. Configuration of singular fibers for the class 2.4

In this picture, the value of  $s_1$  is given by

$$(5.7) \quad s_1 = \frac{(c^2 + 2cd - 2c - d + 1)a + (d-1)(bc-b+d)}{c+d-1}.$$

**Acknowledgement.** *The author would like to express his gratitude to Professor Masoto Kuwata for his helpful comments and beneficial advice. The author also would like to thank Professors Hiroyuki Ito and Ichiro Shimada for their very valuable suggestions. The computer algebra system Maple and Maple Library “Elliptic Surface Calculator” written by Professor Masato Kuwata [6] were used in the calculation for this paper. Furthermore the author would like to thank Professor Matthias Schütt for his profitable comments to previous version of this paper.*

#### REFERENCES

- [1] Sang Yook An, Seog Young Kim, David C. Marshall, Susan H. Marshall, William G. McCallum and Alexander R. Perlis, “Jacobians of genus one curves”, J. Number Theory **90** (2001), no. 2, 304-315.
- [2] I. Connell, Addendum to a paper of K. Harada and M.-L. Lang, “Some elliptic curves arising from the Leech lattice” [J. Algebra **125** (1989), no. 2, 298310], J. Algebra **145** (1992), 463-467.
- [3] R. Kloosterman, “Classification of all Jacobian fibrations on certain  $K3$  surfaces”, J. Math. Soc. Japan **58** (2006), no. 3, 665-690.
- [4] K. Kodaira, “On compact analytic surfaces II”, Ann. of Math. **77**, no.3 (1963), 545-560.
- [5] A. Kumar, “Elliptic fibrations on a generic Jacobian Kummer surface”, arXiv:1105.1715v1.
- [6] M. Kuwata, “Maple Library ‘Elliptic Surface Calculator’”, <http://c-faculty.chuo-u.ac.jp/~kuwata/ESC.php>.
- [7] M. Kuwata and T. Shioda, “Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface”, Algebraic geometry in East Asia-Hanoi (2005), 177-215, Adv. Stud. Pure Math., **50**, Math. Soc. Japan, Tokyo, 2008.
- [8] K. Oguiso, “On Jacobian fibrations on the Kummer surfaces of the product of nonisogenous elliptic curves”, J. Math. Soc. Japan **41** (1989), no. 4, 651-680.
- [9] J. H. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, Graduate Text in Mathematics 151, Springer (1994).
- [10] J. Tate, “Algorithm for determining the type of a singular fiber in an elliptic pencil”, Lecture Note in Math. **476**, Springer, (1975), 33-52.

## 6. WEIERSTRASS EQUATIONS

6.1. **Class 1.1** :  $I_{10} + I_2 + aI_1 + bI_1$  with  $\text{MWG} = \mathbb{Z}^4$ .

$$\begin{aligned}
y^2 = & x^3 - \left( 27s^8 - 108(ad + bc - 2a - 2b + c + d + 1)s^6 + 54(3a^2d^2 + 2abcd \right. \\
& + 3b^2c^2 - 8a^2d - 16abc - 16abd + 14acd + 2ad^2 - 8b^2c + 2bc^2 + 14bcd \\
& + 8a^2 + 24ab - 8ac - 2ad + 8b^2 - 2bc - 8bd + 3c^2 - 6cd + 3d^2 - 8a \\
& - 8b + 2c + 2d + 3)s^4 - 108(a^3d^3 - a^2bcd^2 - ab^2c^2d + b^3c^3 - 2a^3d^2 \\
& - 10a^2bcd - 4a^2bd^2 + 5a^2cd^2 - a^2d^3 - 4ab^2c^2 - 10ab^2cd + 12abc^2d \\
& + 12abcd^2 - 2b^3c^2 - b^2c^3 + 5b^2c^2d + 16a^2bc + 18a^2bd - 8a^2cd - 5a^2d^2 \\
& + 18ab^2c + 16ab^2d - 10abc^2 - 18abcd - 10abd^2 + 5ac^2d - 2acd^2 - ad^3 \\
& - 5b^2c^2 - 8b^2cd - bc^3 - 2bc^2d + 5bcd^2 - 16a^2b + 8a^2d - 16ab^2 - 4abc \\
& - 4abd - 2ac^2 + 8ad^2 + 8b^2c + 8bc^2 - 2bd^2 + c^3 - 3c^2d - 3cd^2 + d^3 \\
& + 14ab + 4ac - 5ad - 5bc + 4bd - c^2 + 4cd - d^2 - 2a - 2b - c - d + 1)s^2 \\
& + 27(a^2d^2 - 2abcd + b^2c^2 + 4abc + 4abd - 2acd - 2ad^2 - 2bc^2 - 2bcd \\
& - 4ab + 2ad + 2bc + c^2 + 2cd + d^2 - 2c - 2d + 1)^2 \Big) x \\
& + \left( 54s^{12} - 324(ad + bc - 2a - 2b + c + d + 1)s^{10} + 162(5a^2d^2 + 6abcd + 5b^2c^2 \right. \\
& - 16a^2d - 24ac - 24abd + 18acd + 6ad^2 - 16b^2c + 6bc^2 + 18bcd + 16a^2 \\
& + 40ab - 16ac - 6ad + 16b^2 - 6bc - 16bd + 5c^2 - 2cd + 5d^2 - 16a - 16b \\
& + 6c + 6d + 5)s^8 - 216(5a^3d^3 + 3a^2bcd^2 + 3ab^2c^2d + 5b^3c^3 - 18a^3d^2 \\
& - 45a^2bcd - 33a^2bd^2 + 30a^2cd^2 + 3a^2d^3 - 33ab^2c^2 - 45ab^2cd + 39abc^2d \\
& + 39abcd^2 - 18b^3c^2 + 3b^2c^3 + 30b^2c^2d + 24a^3d + 72a^2bc + 111a^2bd - 66a^2cd \\
& - 15a^2d^2 + 111ab^2c + 72ab^2d - 45abc^2 - 144abcd - 45abd^2 + 30ac^2d + 6acd^2 \\
& + 3ad^3 + 24b^3c - 15b^2c^2 - 66b^2cd + 3bc^3 + 6bc^2d + 30bcd^2 - 16a^3 - 96a^2b \\
& + 24a^2c + 6a^2d - 96ab^2 + 12abc + 12abd - 18ac^2 + 33acd + 6ad^2 - 16b^3 + 6b^2c + \\
& 24b^2d + 6bc^2 + 33bcd - 18bd^2 + 5c^3 - 12c^2d - 12cd^2 + 5d^3 + 24a^2 + 81ab \\
& - 12ac - 15ad + 24b^2 - 15bc - 12bd + 3c^2 - 3cd + 3d^2 - 18a - 18b + 3c + 3d \\
& + 5)s^6 + 162(5a^4d^4 - 4a^3bcd^3 - 2a^2b^2c^2d^2 - 4ab^3c^3d + 5b^4c^4 - 16a^4d^3 \\
& - 44a^3bcd^2 - 28a^3bd^3 + 32a^3cd^3 - 4a^3d^4 - 40a^2b^2c^2d - 40a^2b^2cd^2 + 44a^2bc^2d^2 \\
& + 56a^2bcd^3 - 28ab^3c^3 - 44ab^3c^2d + 56ab^2c^3d + 44ab^2c^2d^2 - 16b^4c^3 - 4b^3c^4 \\
& + 32b^3c^3d + 16a^4d^2 + 144a^3bcd + 124a^3bd^2 - 80a^3cd^2 - 12a^3d^3 + 136a^2b^2c^2 \\
& + 376a^2b^2cd + 136a^2b^2d^2 - 296a^2bc^2d - 372a^2bcd^2 - 40a^2bd^3 + 94a^2c^2d^2 \\
& - 16a^2cd^3 - 2a^2d^4 + 124ab^3c^2 + 144ab^3cd - 40ab^2c^3 - 372ab^2c^2d - 296ab^2cd^2 \\
& + 44abc^3d + 208abc^2d^2 + 44abcd^3 + 16b^4c^2 - 12b^3c^3 - 80b^3c^2d - 2b^2c^4 \\
& - 16b^2c^3d + 94b^2c^2d^2 - 128a^3bc - 208a^3bd + 64a^3cd + 64a^3d^2 - 416a^2b^2c \\
& - 416a^2b^2d + 144a^2bc^2 + 400a^2bcd + 144a^2bd^2 - 80a^2c^2d - 4a^2cd^2 + 4a^2d^3
\end{aligned}$$

$$\begin{aligned}
& -208ab^3c - 128ab^3d + 144ab^2c^2 + 400ab^2cd + 144ab^2d^2 - 44abc^3 + 44abc^2d \\
& + 44abcd^2 - 44abd^3 + 32ac^3d - 108ac^2d^2 - 4ad^4 + 64b^3c^2 + 64b^3cd + 4b^2c^3 \\
& - 4b^2c^2d - 80b^2cd^2 - 4bc^4 - 108bc^2d^2 + 32bcd^3 + 128a^3b - 64a^3d + 296a^2b^2 \\
& + 32a^2bc + 40a^2bd + 16a^2c^2 + 16a^2cd - 66a^2d^2 + 128ab^3 + 40ab^2c + 32ab^2d \\
& - 76abc^2 - 212abcd - 76abd^2 - 16ac^3 + 20ac^2d + 40acd^2 + 28ad^3 - 64b^3c \\
& - 66b^2c^2 + 16b^2cd + 16b^2d^2 + 28bc^3 + 40bc^2d + 20bcd^2 - 16bd^3 + 5c^4 - 16c^3d \\
& + 30c^2d^2 - 16cd^3 + 5d^4 - 176a^2b - 32a^2c + 64a^2d - 176ab^2 + 28abc + 28abd \\
& + 16ac^2 - 40acd + 4ad^2 + 64b^2c - 32b^2d + 4bc^2 - 40bcd + 16bd^2 - 4c^3 + 12c^2d \\
& + 12cd^2 - 4d^3 + 16a^2 + 92ab + 16ac - 12ad + 16b^2 - 12bc + 16bd - 2c^2 + 8cd \\
& - 2d^2 - 16a - 16b - 4c - 4d + 5)s^4 - 324(a^5d^5 - 3a^4bcd^4 + 2a^3b^2c^2d^3 + 2a^2b^3c^3d^2 \\
& - 3ab^4c^4d + b^5c^5 - 2a^5d^4 - 2a^4bcd^3 + 3a^4cd^4 - 3a^4d^5 + 10a^3b^2c^2d^2 - 6a^3b^2cd^3 \\
& + 2a^3bc^2d^3 + 14a^3bcd^4 - 6a^2b^3c^3d + 10a^2b^3c^2d^2 - 16a^2b^2c^3d^2 - 16a^2b^2c^2d^3 \\
& - 2ab^4c^3d + 14ab^3c^4d + 2ab^3c^3d^2 - 2b^5c^4 - 3b^4c^5 + 3b^4c^4d + 8a^4bcd^2 + 6a^4bd^3 \\
& - 4a^4cd^3 + a^4d^4 + 72a^3b^2c^2d + 74a^3b^2cd^2 - 94a^3bc^2d^2 - 104a^3bcd^3 - 6a^3bd^4 \\
& - 4a^3c^2d^3 - 8a^3cd^4 + 2a^3d^5 + 74a^2b^3c^2d + 72a^2b^3cd^2 - 56a^2b^2c^3d - 114a^2b^2c^2d^2 \\
& - 56a^2b^2cd^3 + 98a^2bc^3d^2 + 88a^2bc^2d^3 - 16a^2bcd^4 + 6ab^4c^3 + 8ab^4c^2d - 6ab^3c^4 \\
& - 104ab^3c^3d - 94ab^3c^2d^2 - 16ab^2c^4d + 88ab^2c^3d^2 + 98ab^2c^2d^3 + b^4c^4 - 4b^4c^3d \\
& + 2b^3c^5 - 8b^3c^4d - 4b^3c^3d^2 - 8a^4bd^2 + 4a^4d^3 - 80a^3b^2c^2 - 224a^3b^2cd - 72a^3b^2d^2 \\
& + 80a^3bc^2d + 248a^3bcd^2 + 80a^3bd^3 + 12a^3c^2d^2 + 30a^3cd^3 + 12a^3d^4 - 72a^2b^3c^2 \\
& - 224a^2b^3cd - 80a^2b^3d^2 + 72a^2b^2c^3 + 156a^2b^2c^2d + 156a^2b^2cd^2 + 72a^2b^2d^3 - 94a^2bc^3d \\
& - 174a^2bc^2d^2 + 44a^2bcd^3 + 10a^2bd^4 - 4a^2c^3d^2 + 6a^2cd^4 + 2a^2d^5 - 8ab^4c^2 + 80ab^3c^3 \\
& + 248ab^3c^2d + 80ab^3cd^2 + 10ab^2c^4 + 44ab^2c^3d - 174ab^2c^2d^2 - 94ab^2cd^3 + 2abc^4d \\
& - 102abc^3d^2 - 102abc^2d^3 + 2abcd^4 + 4b^4c^3 + 12b^3c^4 + 30b^3c^3d + 12b^3c^2d^2 + 2b^2c^5 \\
& + 6b^2c^4d - 4b^2c^2d^3 + 160a^3b^2c + 152a^3b^2d - 160a^3bcd - 154a^3bd^2 - 24a^3cd^2 \\
& - 26a^3d^3 + 152a^2b^3c + 160a^2b^3d - 56a^2b^2c^2 + 4a^2b^2cd - 56a^2b^2d^2 + 8a^2bc^3 \\
& + 134a^2bc^2d - 66a^2bcd^2 - 90a^2bd^3 - 4a^2c^3d - 18a^2c^2d^2 - 42a^2cd^3 - 22a^2d^4 \\
& - 154ab^3c^2 - 160ab^3cd - 90ab^2c^3 - 66ab^2c^2d + 134ab^2cd^2 + 8ab^2d^3 - 2abc^4 \\
& + 72abc^3d + 256abc^2d^2 + 72abcd^3 - 2abd^4 + 3ac^4d + 12ac^3d^2 + 12ac^2d^3 \\
& - 3ad^5 - 26b^3c^3 - 24b^3c^2d - 22b^2c^4 - 42b^2c^3d - 18b^2c^2d^2 - 4b^2cd^3 - 3bc^5 \\
& + 12bc^3d^2 + 12bc^2d^3 + 3bcd^4 - 80a^3b^2 + 80a^3bd + 12a^3d^2 - 80a^2b^3 \\
& - 104a^2b^2c - 104a^2b^2d - 24a^2bc^2 + 14a^2bcd + 142a^2bd^2 + 12a^2c^2d + 48a^2cd^2 \\
& + 42a^2d^3 + 80ab^3c + 142ab^2c^2 + 14ab^2cd - 24ab^2d^2 - 4abc^3 - 178abc^2d \\
& - 178abcd^2 - 4abd^3 - 2ac^4 - 10ac^3d - 12ac^2d^2 + 10acd^3 + 14ad^4 + 12b^3c^2 \\
& + 42b^2c^3 + 48b^2c^2d + 12b^2cd^2 + 14bc^4 + 10bc^3d - 12bc^2d^2 - 10bcd^3 - 2bd^4 \\
& + c^5 - c^4d - 8c^3d^2 - 8c^2d^3 - cd^4 + d^5 + 88a^2b^2 + 24a^2bc - 54a^2bd - 12a^2cd
\end{aligned}$$



$$\begin{aligned}
& -26a^2d^2 - 54ab^2c + 24ab^2d + 24abc^2 + 132abcd + 24abd^2 + 8ac^3 + 12ac^2d \\
& -12acd^2 - 22ad^3 - 26b^2c^2 - 12b^2cd - 22bc^3 - 12bc^2d + 12bcd^2 + 8bd^3 - 3c^4 \\
& + 6c^3d + 18c^2d^2 + 6cd^3 - 3d^4 - 8a^2b + 4a^2d - 8ab^2 - 28abc - 28abd - 12ac^2 \\
& - 6acd + 12ad^2 + 4b^2c + 12bc^2 - 6bcd - 12bd^2 + 2c^3 - 12c^2d - 12cd^2 + 2d^3 \\
& + 10ab + 8ac + ad + bc + 8bd + 2c^2 + 10cd + 2d^2 - 2a - 2b - 3c - 3d + 1)s^2 \\
& + 54(a^2d^2 - 2abcd + b^2c^2 + 4abc + 4abd - 2acd - 2ad^2 - 2bc^2 - 2bcd - 4ab \\
& + 2ad + 2bc + c^2 + 2cd + d^2 - 2c - 2d + 1)^3).
\end{aligned}$$

6.2. **Class 1.2 :**  $I_{10} + I_2 + aII + bI_1$  **with**  $\text{MWG} = \mathbb{Z}^4$ .

$$\begin{aligned}
y^2 = & x^3 - \left( 27(ad - bc)^4t^8 - 108(a^3d^3 - a^2bcd^2 - ab^2c^2d + b^3c^3 - 2a^3d^2 - 2a^2bcd \right. \\
& + 4a^2bd^2 + 4a^2cd^2 - 2a^2d^3 + 4ab^2c^2 - 2ab^2cd - 2abc^2d - 2abcd^2 - 2b^3c^2 \\
& - 2b^2c^3 + 4b^2c^2d - 6a^2bd + 4a^2d^2 - 6ab^2c + 16abcd - 6acd^2 + 4b^2c^2 - 6bc^2d)t^6 \\
& + 54(3a^2d^2 + 2abcd + 3b^2c^2 - 8a^2d + 4abc + 4abd + 4acd - 8ad^2 - 8b^2c \\
& - 8bc^2 + 4bcd + 8a^2 + 4ab - 8ac + 8ad + 8b^2 + 8bc - 8bd + 8c^2 + 4cd + 8d^2 \\
& \left. - 8a - 8b - 8c - 8d + 8)t^4 - 108(ad + bc - 2a - 2b - 2c - 2d + 4)t^2 + 27 \right)x \\
& + \left( 54(ad - bc)^6t^{12} - 324(a^3d^3 - a^2bcd^2 - ab^2c^2d + b^3c^3 - 2a^3d^2 - 2a^2bcd + 4a^2bd^2 \right. \\
& + 4a^2cd^2 - 2a^2d^3 + 4ab^2c^2 - 2ab^2cd - 2abc^2d - 2abcd^2 - 2b^3c^2 - 2b^2c^3 + 4b^2c^2d \\
& - 6a^2bd + 4a^2d^2 - 6ab^2c + 16abcd - 6acd^2 + 4b^2c^2 - 6bc^2d)(ad - bc)^2t^{10} \\
& + 162(5a^4d^4 - 4a^3bcd^3 - 2a^2b^2c^2d^2 - 4ab^3c^3d + 5b^4c^4 - 16a^4d^3 + 4a^3bcd^2 \\
& + 20a^3bd^3 + 20a^3cd^3 - 16a^3d^4 - 8a^2b^2c^2d - 8a^2b^2cd^2 - 8a^2bc^2d^2 + 4a^2bcd^3 \\
& + 20ab^3c^3 + 4ab^3c^2d + 4ab^2c^3d - 8ab^2c^2d^2 - 16b^4c^3 - 16b^3c^4 + 20b^3c^3d + 16a^4d^2 \\
& + 16a^3bcd - 52a^3bd^2 - 40a^3cd^2 + 40a^3d^3 + 40a^2b^2c^2 + 152a^2b^2cd + 40a^2b^2d^2 \\
& - 64a^2bc^2d - 88a^2bcd^2 - 40a^2bd^3 + 40a^2c^2d^2 - 52a^2cd^3 + 16a^2d^4 - 52ab^3c^2 \\
& + 16ab^3cd - 40ab^2c^3 - 88ab^2c^2d - 64ab^2cd^2 + 16abc^3d + 152abc^2d^2 + 16abcd^3 \\
& + 16b^4c^2 + 40b^3c^3 - 40b^3c^2d + 16b^2c^4 - 52b^2c^3d + 40b^2c^2d^2 + 48a^3bd - 40a^3d^2 \\
& - 96a^2b^2c - 96a^2b^2d - 16a^2bcd + 104a^2bd^2 + 104a^2cd^2 - 40a^2d^3 + 48ab^3c \\
& + 104ab^2c^2 - 16ab^2cd - 16abc^2d - 16abcd^2 - 96ac^2d^2 + 48acd^3 - 40b^3c^2 - 40b^2c^3 \\
& + 104b^2c^2d + 48bc^3d - 96bc^2d^2 + 72a^2b^2 - 96a^2bd + 40a^2d^2 - 96ab^2c + 160abcd \\
& - 96acd^2 + 40b^2c^2 - 96bc^2d + 72c^2d^2)t^8 - 216(5a^3d^3 + 3a^2bcd^2 + 3ab^2c^2d \\
& + 5b^3c^3 - 18a^3d^2 - 3a^2bcd + 9a^2bd^2 + 9a^2cd^2 - 18a^2d^3 + 9ab^2c^2 - 3ab^2cd \\
& - 3abc^2d - 3abcd^2 - 18b^3c^2 - 18b^2c^3 + 9b^2c^2d + 24a^3d - 12a^2bc - 15a^2bd \\
& - 24a^2cd + 48a^2d^2 - 15ab^2c - 12ab^2d - 24abc^2 - 60abcd - 24abd^2 - 12ac^2d \\
& - 15acd^2 + 24ad^3 + 24b^3c + 48b^2c^2 - 24b^2cd + 24bc^3 - 15bc^2d - 12bcd^2 - 16a^3 \\
& \left. - 12a^2b + 24a^2c - 36a^2d - 12ab^2 + 96abc + 96abd + 24ac^2 + 96acd - 36ad^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -16b^3 - 36b^2c + 24b^2d - 36bc^2 + 96bcd + 24bd^2 - 16c^3 - 12c^2d - 12cd^2 \\
& -16d^3 + 24a^2 - 24ab - 96ac - 36ad + 24b^2 - 36bc - 96bd + 24c^2 - 24cd \\
& + 24d^2 + 24a + 24b + 24c + 24d - 16)t^6 + 162(5a^2d^2 + 6abcd + 5b^2c^2 \\
& - 16a^2d - 4abc - 4abd - 4acd - 16ad^2 - 16b^2c - 16bc^2 - 4bcd + 16a^2 \\
& + 20ab + 8ac + 40ad + 16b^2 + 40bc + 8bd + 16c^2 + 20cd + 16d^2 - 40a \\
& - 40b - 40c - 40d + 40)t^4 - 324(ad + bc - 2a - 2b - 2c - 2d + 4)t^2 + 54).
\end{aligned}$$

6.3. **Class 1.3 :  $2I_6 + aII + bI_1$  with  $\text{MWG} = \mathbb{Z}^4$ .**

$$\begin{aligned}
y^2 = x^3 - & \left( 27t^8 + (216bc + 216 + 216ad + 648ac - 432c - 108b - 108d - 432a)t^6 \right. \\
& + (432 - 432a - 432d - 432b - 432c + 432bc + 432ad + 108bd - 432ad^2 - 432b^2c \\
& + 432a^2 + 432d^2a^2 - 432da^2 + 216abc432ac + 216cd + 216bcd + 216abd + 216ab \\
& - 432bc^2 + 432b^2c^2 + 432c^2 + 162d^2 + 162b^2 - 432abcd + 216acd)t^4 \\
& - 108(b-d)(b^2 - 2b^2c - 2b + 4bc + 4abd - 2ab - 4bcd - 4ad + 2cd + 2d + 2ad^2 \\
& \left. - d^2)t^2 + 27(b-d)^4 \right)x \\
& + \left( 54t^{12} + (1944ac + 648bc + 648ad - 324b - 324d - 1296a + 648 - 1296c)t^{10} \right. \\
& + (2592 - 6480a - 2592d - 2592b - 6480c - 15552a^2c + 7776abc^2 + 6480bc \\
& + 6480ad + 972bd - 2592ad^2 - 2592b^2c + 6480a^2 + 2592a^2d^2 - 6480a^2d - 8424abc \\
& - 15552ac^2 + 11664a^2c^2 + 16848ac + 3240cd - 648bcd - 648abd + 3240ba - 6480bc^2 \\
& + 2592b^2c^2 + 6480c^2 + 810d^2 + 810b^2 + 1296abcd - 8424acd + 7776a^2cd)t^8 \\
& + (3456 - 5184a - 5184d - 5184b - 5184c + 2592a^2bc + 2592ab^2c^2 - 5184a^2c + 5184abc^2 \\
& + 7776bc + 7776ad + 2592bd + 2592a^2b - 10368ad^2 - 10368b^2c - 1944cd^2 + 3888ad^3 \\
& + 3888b^3c - 648bd^2 - 648b^2d + 3240ab^2c - 5184a^2 + 3456a^3 + 7776a^2d^2 - 5184a^3d^2 \\
& - 5184a^3d - 10368bc^2d + 7776a^2d + 2592b^2c^2d + 2592a^2bd^2 + 5184ab^2cd + 5184abcd^2 \\
& - 5184ab^2c^2d - 5184a^2bcd^2 + 5184abc^2d + 5184a^2bcd - 20736cba - 5184ac^2 - 5184b^3c^2 \\
& + 3456a^3d^3 - 5184a^2d^3 + 3456b^3c^3 - 5184b^2c^3 - 5184bc^3 + 20736ac + 2592c^2d + 5184cd \\
& - 10368a^2bd + 1944bcd + 1944abd + 5184ab + 7776c^2b + 7776c^2b^2 - 5184c^2 + 3888d^2 \\
& + 3888b^2 + 3456c^3 - 1080b^3 - 1080d^3 + 3888abcd + 2592ac^2d - 20736acd + 2592a^2cd^2 \\
& + 3240d^2ac + 5184a^2cd - 1944ab^2 + 648b^2cd + 648abd^2 - 1944bcd^2 - 1944ab^2d)t^6 \\
& + (2592b^4c^2 - 2592ad^4 + 2592a^2d^4 - 2592b^4c - 5184bd - 324b^2d^2 - 6480ad^2 \\
& - 6480b^2c + 1296cd^2 + 6480ad^3 + 6480b^3c + 2592bd^2 + 2592b^2d - 6480ab^2c \\
& - 648bd^3 - 648b^3d + 6480d^2a^2 + 2592bc^2d - 10368b^2c^2d - 10368a^2bd^2 + 20088ab^2cd \\
& + 20088abcd^2 - 10368ab^2cd^2 - 6480b^3c^2 + 2592a^2b^2 - 648ab^3 - 6480a^2d^3 + 2592c^2d^2 \\
& - 648cd^3 - 6480bc^2d^2 + 3240ab^3c - 6480a^2b^2d + 3240ab^3d - 1296ab^2d^2 + 648abd^3 \\
& \left. + 648b^3cd - 1296b^2cd^2 + 3240bcd^3 + 2592b^3c^2d + 2592a^2bd^3 + 3240acd^3 + 6480b^2c^2d^2 \right)
\end{aligned}$$

$$\begin{aligned}
& + 6480a^2b^2d^2 + 2592a^2bd + 5184bcd + 5184abd + 6480c^2b^2 - 6480abcd^3 - 6480ab^3cd \\
& + 2592d^2 + 2592b^2 + 810b^4 - 2592b^3 - 2592d^3 + 810d^4 - 10368abcd - 6480acd^2 \\
& + 1296ab^2 + 1944b^2cd + 1944abd^2 - 7776bcd^2 - 7776ab^2d)t^4 \\
& - 324(b-d)^3(b^2 - 2b^2c - 2b + 4bc + 4abd - 2ab - 4bcd - 4ad + 2cd + 2d + 2ad^2 \\
& - d^2)t^2 + 54(b-d)^6).
\end{aligned}$$

6.4. **Class 1.4** :  $IV^* + I_4 + aII + bI_1$  with  $MWG = \mathbb{Z}^5$ .

$$\begin{aligned}
y^2 = & x^3 - \left( 216(2a^2d^2 - 2abcd + 2b^2c^2 - 2a^2d + abc + abd + acd - 2ad^2 - 2b^2c \right. \\
& - 2bc^2 + bcd + 2a^2 + ab - 2ac + 2ad + 2b^2 + 2bc - 2bd + 2c^2 + cd + 2d^2 \\
& - 2a - 2b - 2c - 2d + 2)s^4 - 216(2a^3b^2c^2d + a^3b^2cd^2 - a^3b^2d^3 - a^3bc^2d^2 \\
& - 4a^3bcd^3 - a^3c^2d^3 - a^2b^3c^3 + a^2b^3c^2d + 2a^2b^3cd^2 - a^2b^2c^3d + 4a^2b^2c^2d^2 \\
& - a^2b^2cd^3 + 2a^2bc^3d^2 + a^2bc^2d^3 - 4ab^3c^3d - ab^3c^2d^2 + ab^2c^3d^2 + 2ab^2c^2d^3 \\
& - b^3c^3d^2 - a^3b^2c^2 - 3a^3b^2cd + a^3b^2d^2 - a^3bc^2d + 6a^3bcd^2 + 3a^3bd^3 + 2a^3c^2d^2 \\
& + 3a^3cd^3 + a^2b^3c^2 - 3a^2b^3cd - a^2b^3d^2 + 2a^2b^2c^3 - 7a^2b^2c^2d - 7a^2b^2cd^2 \\
& + 2a^2b^2d^3 - a^2bc^3d - 7a^2bc^2d^2 + 6a^2bcd^3 - a^2c^3d^2 + a^2c^2d^3 + 3ab^3c^3 \\
& + 6ab^3c^2d - ab^3cd^2 + 6ab^2c^3d - 7ab^2c^2d^2 - ab^2cd^3 - 3abc^3d^2 - 3abc^2d^3 \\
& + 3b^3c^3d + 2b^3c^2d^2 + b^2c^3d^2 - b^2c^2d^3 + 2a^3b^2c + a^3b^2d - 2a^3bcd \\
& - 6a^3bd^2 - 3a^3cd^2 + a^2b^3c + 2a^2b^3d - 2a^2b^2c^2 + 10a^2b^2cd - 2a^2b^2d^2 \\
& + 10a^2bc^2d - 3a^2bd^3 - 2a^2c^2d^2 - 6a^2cd^3 - 6ab^3c^2 - 2ab^3cd - 3ab^2c^3 \\
& + 10ab^2cd^2 - 2abc^3d + 10abc^2d^2 - 2abcd^3 + 2ac^3d^2 + ac^2d^3 - 3b^3c^2d \\
& - 6b^2c^3d - 2b^2c^2d^2 + bc^3d^2 + 2bc^2d^3 - a^3b^2 + 3a^3bd - a^2b^3 - 2a^2b^2c \\
& - 2a^2b^2d - 6a^2bcd + 6a^2bd^2 + 6a^2cd^2 + 3ab^3c + 6ab^2c^2 - 6ab^2cd \\
& - 6abc^2d - 6abcd^2 - 2ac^2d^2 + 3acd^3 + 6b^2c^2d + 3bc^3d - 2bc^2d^2 - c^3d^2 \\
& - c^2d^3 + 2a^2b^2 - 3a^2bd - 3ab^2c + 8abcd - 3acd^2 - 3bc^2d + 2c^2d^2)s^2 \\
& \left. + 27(abc + abd - acd - bcd - ab + cd)^4 \right)x \\
& + \left( 11664s^8 + 864(4a^3d^3 - 6a^2bcd^2 - 6ab^2c^2d + 4b^3c^3 - 6a^3d^2 + 6a^2bcd \right. \\
& + 3a^2bd^2 + 3a^2cd^2 - 6a^2d^3 + 3ab^2c^2 + 6ab^2cd + 6abc^2d + 6abcd^2 - 6b^3c^2 - 6b^2c^3 \\
& + 3b^2c^2d - 6a^3d + 3a^2bc - 12a^2bd + 6a^2cd + 9a^2d^2 - 12ab^2c + 3ab^2d + 6abc^2 \\
& + 36abcd + 6abd^2 + 3ac^2d - 12acd^2 - 6ad^3 - 6b^3c + 9b^2c^2 + 6b^2cd - 6bc^3 \\
& - 12bc^2d + 3bcd^2 + 4a^3 + 3a^2b - 6a^2c + 9a^2d + 3ab^2 - 24abc - 24abd \\
& - 6ac^2 - 24acd + 9ad^2 + 4b^3 + 9b^2c - 6b^2d + 9bc^2 - 24bcd - 6bd^2 + 4c^3 \\
& + 3c^2d + 3cd^2 + 4d^3 - 6a^2 + 6ab + 24ac + 9ad - 6b^2 + 9bc + 24bd - 6c^2 \\
& + 6cd - 6d^2 - 6a - 6b - 6c - 6d + 4)s^6 + 648(10a^4b^2c^2d^2 - 4a^4b^2cd^3 \\
& \left. + 4a^4b^2d^4 + 4a^4bc^2d^3 + 16a^4bcd^4 + 4a^4c^2d^4 - 10a^3b^3c^3d + 16a^3b^3c^2d^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -10a^3b^3cd^3 - 16a^3b^2c^3d^2 - 24a^3b^2c^2d^3 + 4a^3b^2cd^4 - 10a^3bc^3d^3 - 4a^3bc^2d^4 \\
& + 4a^2b^4c^4 - 4a^2b^4c^3d + 10a^2b^4c^2d^2 + 4a^2b^3c^4d - 24a^2b^3c^3d^2 - 16a^2b^3c^2d^3 \\
& + 10a^2b^2c^4d^2 + 16a^2b^2c^3d^3 + 10a^2b^2c^2d^4 + 16ab^4c^4d + 4ab^4c^3d^2 - 4ab^3c^4d^2 \\
& - 10ab^3c^3d^3 + 4b^4c^4d^2 - 10a^4b^2c^2d - 4a^4b^2cd^2 - 6a^4b^2d^3 - 16a^4bc^2d^2 \\
& - 32a^4bcd^3 - 12a^4bd^4 - 10a^4c^2d^3 - 12a^4cd^4 + 5a^3b^3c^3 - a^3b^3c^2d - a^3b^3cd^2 \\
& + 5a^3b^3d^3 + 31a^3b^2c^3d + 16a^3b^2c^2d^2 + 39a^3b^2cd^3 - 10a^3b^2d^4 + 31a^3bc^3d^2 \\
& + 39a^3bc^2d^3 - 32a^3bcd^4 + 5a^3c^3d^3 - 6a^3c^2d^4 - 6a^2b^4c^3 - 4a^2b^4c^2d - 10a^2b^4cd^2 \\
& - 10a^2b^3c^4 + 39a^2b^3c^3d + 16a^2b^3c^2d^2 + 31a^2b^3cd^3 - 16a^2b^2c^4d \\
& + 16a^2b^2c^3d^2 + 16a^2b^2c^2d^3 - 16a^2b^2cd^4 - 10a^2bc^4d^2 - a^2bc^3d^3 - 4a^2bc^2d^4 \\
& - 12ab^4c^4 - 32ab^4c^3d - 16ab^4c^2d^2 - 32ab^3c^4d + 39ab^3c^3d^2 + 31ab^3c^2d^3 \\
& - 4ab^2c^4d^2 - ab^2c^3d^3 - 10ab^2c^2d^4 - 12b^4c^4d - 10b^4c^3d^2 - 6b^3c^4d^2 + 5b^3c^3d^3 \\
& + 4a^4b^2c^2 + 16a^4b^2cd + 4a^4b^2d^2 + 4a^4bc^2d + 8a^4bcd^2 + 12a^4bd^3 + 10a^4c^2d^2 \\
& + 36a^4cd^3 + 18a^4d^4 - 5a^3b^3c^2 + 16a^3b^3cd - 5a^3b^3d^2 - 10a^3b^2c^3 - 28a^3b^2c^2d \\
& - 56a^3b^2cd^2 - 6a^3b^2d^3 - 16a^3bc^3d - 21a^3bc^2d^2 + 40a^3bcd^3 + 36a^3bd^4 \\
& - 10a^3c^3d^2 - 6a^3c^2d^3 + 12a^3cd^4 + 4a^2b^4c^2 + 16a^2b^4cd + 4a^2b^4d^2 - 6a^2b^3c^3 \\
& - 56a^2b^3c^2d - 28a^2b^3cd^2 - 10a^2b^3d^3 + 10a^2b^2c^4 - 21a^2b^2c^3d + 12a^2b^2c^2d^2 \\
& - 21a^2b^2cd^3 + 10a^2b^2d^4 + 4a^2bc^4d - 28a^2bc^3d^2 - 56a^2bc^2d^3 + 8a^2bcd^4 \\
& + 4a^2c^4d^2 - 5a^2c^3d^3 + 4a^2c^2d^4 + 12ab^4c^3 + 8ab^4c^2d + 4ab^4cd^2 + 36ab^3c^4 \\
& + 40ab^3c^3d - 21ab^3c^2d^2 - 16ab^3cd^3 + 8ab^2c^4d - 56ab^2c^3d^2 - 28ab^2c^2d^3 \\
& + 4ab^2cd^4 + 16abc^4d^2 + 16abc^3d^3 + 16abc^2d^4 + 18b^4c^4 + 36b^4c^3d + 10b^4c^2d^2 \\
& + 12b^3c^4d - 6b^3c^3d^2 - 10b^3c^2d^3 + 4b^2c^4d^2 - 5b^2c^3d^3 + 4b^2c^2d^4 - 8a^4b^2c \\
& - 6a^4b^2d + 8a^4bcd + 12a^4bd^2 - 24a^4cd^2 - 36a^4d^3 - 5a^3b^3c - 5a^3b^3d \\
& + 10a^3b^2c^2 + 3a^3b^2cd + 32a^3b^2d^2 + 8a^3bc^2d + 20a^3bcd^2 - 36a^3bd^3 + 6a^3c^2d^2 \\
& - 36a^3cd^3 - 36a^3d^4 - 6a^2b^4c - 8a^2b^4d + 32a^2b^3c^2 + 3a^2b^3cd + 10a^2b^3d^2 \\
& + 6a^2b^2c^3 + 16a^2b^2c^2d + 16a^2b^2cd^2 + 6a^2b^2d^3 + 8a^2bc^3d + 16a^2bc^2d^2 \\
& + 20a^2bcd^3 - 24a^2bd^4 + 10a^2c^3d^2 + 32a^2c^2d^3 + 12a^2cd^4 + 12ab^4c^2 \\
& + 8ab^4cd - 36ab^3c^3 + 20ab^3c^2d + 8ab^3cd^2 - 24ab^2c^4 + 20ab^2c^3d + 16ab^2c^2d^2 \\
& + 8ab^2cd^3 + 8abc^4d + 3abc^3d^2 + 3abc^2d^3 + 8abcd^4 - 8ac^4d^2 - 5ac^3d^3 - 6ac^2d^4 \\
& - 36b^4c^3 - 24b^4c^2d - 36b^3c^4 - 36b^3c^3d + 6b^3c^2d^2 + 12b^2c^4d + 32b^2c^3d^2 \\
& + 10b^2c^2d^3 - 6bc^4d^2 - 5bc^3d^3 - 8bc^2d^4 + 4a^4b^2 - 12a^4bd + 18a^4d^2 + 5a^3b^3 \\
& + 10a^3b^2c - 6a^3b^2d - 28a^3bcd - 36a^3bd^2 + 24a^3cd^2 + 72a^3d^3 + 4a^2b^4 - 6a^2b^3c \\
& + 10a^2b^3d - 32a^2b^2c^2 + 15a^2b^2cd - 32a^2b^2d^2 + 32a^2bc^2d - 8a^2bcd^2 + 24a^2bd^3 \\
& - 32a^2c^2d^2 - 36a^2cd^3 + 18a^2d^4 - 12ab^4c - 36ab^3c^2 - 28ab^3cd + 24ab^2c^3 \\
& - 8ab^2c^2d + 32ab^2cd^2 - 28abc^3d + 15abc^2d^2 - 28abcd^3 + 10ac^3d^2 - 6ac^2d^3 \\
& - 12acd^4 + 18b^4c^2 + 72b^3c^3 + 24b^3c^2d + 18b^2c^4 - 36b^2c^3d - 32b^2c^2d^2
\end{aligned}$$

$$\begin{aligned}
& -12bc^4d - 6bc^3d^2 + 10bc^2d^3 + 4c^4d^2 + 5c^3d^3 + 4c^2d^4 - 10a^3b^2 + 36a^3bd \\
& - 36a^3d^2 - 10a^2b^3 + 6a^2b^2c + 6a^2b^2d - 20a^2bcd + 24a^2bd^2 + 24a^2cd^2 \\
& - 36a^2d^3 + 36ab^3c + 24ab^2c^2 - 20ab^2cd - 20abc^2d - 20abcd^2 + 6ac^2d^2 \\
& + 36acd^3 - 36b^3c^2 - 36b^2c^3 + 24b^2c^2d + 36bc^3d + 6bc^2d^2 - 10c^3d^2 - 10c^2d^3 \\
& + 10a^2b^2 - 24a^2bd + 18a^2d^2 - 24ab^2c + 40abcd - 24acd^2 + 18b^2c^2 - 24bc^2d \\
& + 10c^2d^2)s^4 - 648(2a^3b^2c^2d + a^3b^2cd^2 - a^3b^2d^3 - a^3bc^2d^2 - 4a^3bcd^3 - a^3c^2d^3 \\
& - a^2b^3c^3 + a^2b^3c^2d + 2a^2b^3cd^2 - a^2b^2c^3d + 4a^2b^2c^2d^2 - a^2b^2cd^3 + 2a^2bc^3d^2 \\
& + a^2bc^2d^3 - 4ab^3c^3d - ab^3c^2d^2 + ab^2c^3d^2 + 2ab^2c^2d^3 - b^3c^3d^2 - a^3b^2c^2 \\
& - 3a^3b^2cd + a^3b^2d^2 - a^3bc^2d + 6a^3bcd^2 + 3a^3bd^3 + 2a^3c^2d^2 + 3a^3cd^3 + a^2b^3c^2 \\
& - 3a^2b^3cd - a^2b^3d^2 + 2a^2b^2c^3 - 7a^2b^2c^2d - 7a^2b^2cd^2 + 2a^2b^2d^3 - a^2bc^3d \\
& - 7a^2bc^2d^2 + 6a^2bcd^3 - a^2c^3d^2 + a^2c^2d^3 + 3ab^3c^3 + 6ab^3c^2d - ab^3cd^2 + 6ab^2c^3d \\
& - 7ab^2c^2d^2 - ab^2cd^3 - 3abc^3d^2 - 3abc^2d^3 + 3b^3c^3d + 2b^3c^2d^2 + b^2c^3d^2 - b^2c^2d^3 \\
& + 2a^3b^2c + a^3b^2d - 2a^3bcd - 6a^3bd^2 - 3a^3cd^2 + a^2b^3c + 2a^2b^3d - 2a^2b^2c^2 \\
& + 10a^2b^2cd - 2a^2b^2d^2 + 10a^2bc^2d - 3a^2bd^3 - 2a^2c^2d^2 - 6a^2cd^3 - 6ab^3c^2 \\
& - 2ab^3cd - 3ab^2c^3 + 10ab^2cd^2 - 2abc^3d + 10abc^2d^2 - 2abcd^3 + 2ac^3d^2 + ac^2d^3 \\
& - 3b^3c^2d - 6b^2c^3d - 2b^2c^2d^2 + bc^3d^2 + 2bc^2d^3 - a^3b^2 + 3a^3bd - a^2b^3 - 2a^2b^2c \\
& - 2a^2b^2d - 6a^2bcd + 6a^2bd^2 + 6a^2cd^2 + 3ab^3c + 6ab^2c^2 - 6ab^2cd - 6abc^2d \\
& - 6abcd^2 - 2ac^2d^2 + 3acd^3 + 6b^2c^2d + 3bc^3d - 2bc^2d^2 - c^3d^2 - c^2d^3 + 2a^2b^2 \\
& - 3a^2bd - 3ab^2c + 8abcd - 3acd^2 - 3bc^2d + 2c^2d^2)(abc + abd - acd - bcd \\
& - ab + cd)^2s^2 + 54(abc + abd - acd - bcd - ab + cd)^6).
\end{aligned}$$

6.5. **Class 2.1** :  $\text{II}^* + 6\text{I}_2 + 2\text{I}_1$  with  $\text{MWG} = \{0\}$ .

$$\begin{aligned}
y^2 &+ \left(2s^2 + 4b - 4ac - 2ad - 2 - 2d + 2c + 4a - 2cb\right)xy - \left(4c(a-1)s^4 - 4(3a^2c^2 - a^2d \right. \\
&- 2acd - c + a^2 + c^2 + abcd + ad^2 - bc^2 + 5ac + 2a^2cd - 4a^2c - 3abc - 2bcd - a + 2abc^2 \\
&+ cd - 4ac^2 + 2bc)s^2 + 4a(c+d-1)(1-3a+a^2cd+abcd+6ac+ad^2-a^2d-3ac^2 \\
&+ 2cd+3abc^2-2bcd-4a^2c+2a^2c^2-5abc+b^2c^2-2bad+2ad+2bc-2d-2c-2acd \\
&+ 2a^2-2bc^2+c^2+d^2+2ab)\Big)y \\
&= x^3 + \left(2c(1-a)s^2 - 2a + 2a^2cd + 2abcd + 4ac + 2ad^2 - 2a^2d - 2ac^2 + 4cd + 4b^2c + 2abc^2 \right. \\
&- 4bcd - 4a^2c + 2a^2c^2 - 2abc - 4bad + 4bd + 4ad - 4bc - 4b^2 - 4d + 4b - 4acd + 2a^2\Big)x^2 \\
&- \left(4c^2(a-1)^2s^4 - 8ac(c+d-1)(a-1)(ac-a-c+cb-d+1)s^2 + 4a^2(c+d-1)^2(ac \right. \\
&- a-c+cb-d+1)^2\Big)x - 8\left(c(1-a)s^2 - a + a^2cd + abcd + 2ac + ad^2 - a^2d - ac^2 + 2cd \right. \\
&+ 2b^2c + abc^2 - 2bcd - 2a^2c + a^2c^2 - abc - 2bad + 2bd + 2ad - 2bc - 2b^2 - 2d + 2b - 2acd \\
&+ a^2\Big)\left(c^2(a-1)^2s^4 - 2ac(c+d-1)(a-1)(ac-a-c+cb-d+1)s^2 + a^2(c+d-1)^2(ac \right. \\
&- a-c+cb-d+1)^2\Big).
\end{aligned}$$

6.6. **Class 2.2** :  $\text{III}^* + 7\text{I}_2 + \text{I}_1$  with  $\text{MWG} = \mathbb{Z}/2\mathbb{Z}$ .

$$\begin{aligned}
y^2 &= x^3 - 9\left((1-2a-2b+c+bc+d+ad)s^2 + (4bc^2b+c^2b^2d-ab^2c^2+ab^2c-4b^2c^2 \right. \\
&+ 2b^2c^3-abcd^2-4abcd-abc^2d+a^2bcd+ab^2cd+cd^2-cd+2bc^3+2ad-2bc^2+2b^2c \\
&- 2b^2cd-bcd+bcd^2-ac^2d-acd+2a^2cd-a^2cd^2+2a^2d^2-2ad^2-2a^2d+c^2d-3abc^2 \\
&+ 4abc-ab+abd^2+a^2bd-a^2bd^2)s - bc(c+d-1)(a-c)(ad-c-bc-d+1)(ad-bc \\
&+ b-d)\Big)x^2 \\
&+ 81s(s+acd+ad^2-bc^2-bcd-ad+bc)(s+abd+acd-b^2c-bc^2-ad+b^2+2bc \\
&- bd-cd-b+d)(s+abcd-bc^2-bcd+bc)(s-a+a^2+c-ac+abc-bc^2+ad \\
&- a^2d-cd+acd)x.
\end{aligned}$$

6.7. **Class 2.3** :  $\text{III}^* + \text{I}_0^* + 3\text{I}_2 + 3\text{I}_1$  with  $\text{MWG} = \{0\}$ .

$$\begin{aligned}
y^2 &= x^3 + 27s^2\left(3(a-1)s^3 + (a+d+b+c+abcd-a^2d^2-b^2c^2-a^2-b^2-c^2-d^2-ad-bc \right. \\
&+ a^2d+7abc+ad^2+bc^2+b^2c+7cd+bd-8ab+ac+7abd-8acd-8bcd-1)s^2 + (7bc^2d^2 \\
&- b^3c^2d-2acd^2+9bc^2d+b^3c^2d-2a^2cd^2+3a^2bd^2+2a^2cd^3-2a^2bd^3+7ac^2d^2-3b^2c^2d^2 \\
&- 5b^2c^2d-3bc^3d-acd^3-3abc^2d^2+3ab^2c^2d+3abcd^3-a^2bcd^2-8abcd^2+2ab^2cd+3ab^2cd^2 \\
&- 8abc^2d+5abcd-a^2d^3+3a^2d^2+3b^3c^2+3b^2c^3-2b^3c^3-3b^2c^2-4c^2d^2+ad+bc-2a^2d \\
&+ 4abc-2ac^2d-2abd^3+4ab^2d^2+4ab^2c^2-b^2cd^2+2a^2cd-a^2bd-7ab^2c-abc^2-7ab^2d \\
&- 3ad^2+2ad^3+cd^2+cd-3c^2d-b^3c+2c^3d-bc^3-a^2bcd+3ab^2-3ab-abd^2+6abd+acd \\
&- 7bcd+7b^2cd-b^3cd)s + (a^2bc^2d^3-a^2b^2c^2d^2-a^2b^2cd^3-a^2b^2d^4-a^2bcd^4-a^2c^2d^4
\end{aligned}$$

$$\begin{aligned}
& + 2ab^3c^3d + 2ab^3c^2d^2 + 2ab^3cd^3 - 2ab^2c^3d^2 + 2ab^2c^2d^3 + 2abc^3d^3 - b^4c^4 - b^4c^3d - b^4c^2d^2 \\
& + b^3c^4d - b^3c^3d^2 - b^2c^4d^2 + a^2b^2cd^2 + 2a^2b^2d^3 + a^2bc^2d^2 + 4a^2bcd^3 + 2a^2bd^4 + a^2c^2d^3 + a^2cd^4 \\
& - 3ab^3c^2d - 3ab^3cd^2 - 3ab^2c^3d - 4ab^2c^2d^2 - 5ab^2cd^3 + abc^3d^2 - 5abc^2d^3 - 2ac^3d^3 + 2b^4c^3 \\
& + b^4c^2d + 2b^3c^4 + 3b^3c^2d^2 - 2b^2c^4d + 4b^2c^3d^2 + 2bc^4d^2 - a^2b^2d^2 - 3a^2bcd^2 - 4a^2bd^3 - a^2c^2d^2 \\
& - 3a^2cd^3 - a^2d^4 + ab^3cd + 5ab^2c^2d + 7ab^2cd^2 + abc^3d + 4abc^2d^2 + 4abcd^3 + ac^3d^2 + 3ac^2d^3 \\
& - b^4c^2 - 4b^3c^3 - b^3c^2d - b^2c^4 + 3b^2c^3d - 4b^2c^2d^2 + bc^4d - 5bc^3d^2 - c^4d^2 + 2a^2bd^2 + 2a^2cd^2 \\
& + 2a^2d^3 - 2ab^2cd - 2abc^2d - 5abcd^2 - 2ac^2d^2 - acd^3 + 2b^3c^2 + 2b^2c^3 - b^2c^2d - 2bc^3d + 3bc^2d^2 \\
& + 2c^3d^2 - a^2d^2 + abcd + acd^2 - b^2c^2 + bc^2d - c^2d^2) \Big) x \\
& + 27s^3 \Big( (18a^2 - 9a^2d - 9abc + 18ab - 9ac + 27ad - 18bc - 27a + 9b + 9c - 18d + 9)s^4 + (2a^3d^3 \\
& - 3a^2bcd^2 - 3ab^2c^2d + 2b^3c^3 - 3a^3d^2 + 3a^2bcd - 30a^2bd^2 + 33a^2cd^2 - 3a^2d^3 - 30ab^2c^2 + 3ab^2cd \\
& + 3abc^2d + 66abcd^2 - 3b^3c^2 - 3b^2c^3 - 30b^2c^2d - 3a^3d + 33a^2bc + 57a^2bd - 60a^2cd - 27a^2d^2 \\
& + 57ab^2c + 33ab^2d + 3abc^2 - 45abcd + 3abd^2 + 33ac^2d - 69acd^2 - 3ad^3 - 3b^3c - 27b^2c^2 + 3b^2cd \\
& - 3bc^3 + 57bc^2d - 30bcd^2 + 2a^3 - 30a^2b - 3a^2c + 36a^2d - 30ab^2 - 75abc - 75abd - 3ac^2 \\
& + 51acd + 36ad^2 + 2b^3 + 36b^2c - 3b^2d + 36bc^2 - 12bcd - 3bd^2 + 2c^3 - 30c^2d + 33cd^2 + 2d^3 \\
& - 3a^2 + 66ab + 12ac - 27ad - 3b^2 - 27bc + 12bd - 3c^2 + 3cd - 3d^2 - 3a - 3b - 3c - 3d + 2) s^3 \\
& + (3a^3bcd^3 + 6a^3bd^4 - 6a^3cd^4 + 3a^2b^2c^2d^2 - 12a^2b^2cd^3 + 12a^2bc^2d^3 - 9a^2bcd^4 - 12ab^3c^3d \\
& + 3ab^3c^2d^2 - 3ab^2c^3d^2 + 27ab^2c^2d^3 + 6b^4c^4 + 3b^4c^3d - 3b^3c^4d - 18b^3c^3d^2 - 12a^3bcd^2 \\
& - 12a^3bd^3 + 9a^3cd^3 + 3a^3d^4 + 27a^2b^2c^2d - 3a^2b^2cd^2 - 33a^2b^2d^3 - 3a^2bc^2d^2 + 3a^2bcd^3 \\
& + 3a^2bd^4 - 45a^2c^2d^3 + 6a^2cd^4 - 33ab^3c^3 - 3ab^3c^2d + 27ab^3cd^2 + 39ab^2c^3d + 102ab^2c^2d^2 \\
& + 45ab^2cd^3 + 33abc^3d^2 - 99abc^2d^3 - 9abcd^4 - 12b^4c^3 - 12b^4c^2d - 12b^3c^4 - 42b^3c^3d \\
& - 3b^3c^2d^2 - 3b^2c^4d + 57b^2c^3d^2 - 18b^2c^2d^3 + 3a^3bcd + 3a^3bd^2 + 9a^3cd^2 + 3a^3d^3 + 15a^2b^2c^2 \\
& + 39a^2b^2cd + 72a^2b^2d^2 - 93a^2bc^2d - 69a^2bcd^2 + 9a^2bd^3 + 72a^2c^2d^2 + 66a^2cd^3 + 6a^2d^4 \\
& + 72ab^3c^2 + 39ab^3cd + 15ab^3d^2 + 27ab^2c^3 - 165ab^2c^2d - 252ab^2cd^2 - 15ab^2d^3 - 3abc^3d \\
& - 42abc^2d^2 + 36abcd^3 + 6abd^4 - 45ac^3d^2 + 57ac^2d^3 + 3acd^4 + 3b^4c^2 + 3b^4cd + 6b^3c^3 \\
& + 54b^3c^2d - 12b^3cd^2 + 3b^2c^4 + 102b^2c^3d - 3b^2c^2d^2 + 3b^2cd^3 + 12bc^4d - 72bc^3d^2 + 33bc^2d^3 \\
& + 3a^3bd - 6a^3cd - 12a^3d^2 - 33a^2b^2c - 57a^2b^2d + 3a^2bc^2 + 105a^2bcd + 6a^2bd^2 + 9a^2c^2d \\
& - 78a^2cd^2 - 30a^2d^3 - 57ab^3c - 33ab^3d - 90ab^2c^2 + 90ab^2cd + 12ab^2d^2 - 12abc^3 + 132abc^2d \\
& + 165abcd^2 + 3abd^3 + 9ac^3d - 6ac^2d^2 - 54acd^3 - 6ad^4 + 3b^4c + 21b^3c^2 + 3b^3cd + 21b^2c^3 \\
& - 102b^2c^2d + 18b^2cd^2 + 3bc^4 - 75bc^3d + 33bc^2d^2 - 6c^4d + 33c^3d^2 - 15c^2d^3 + 6a^3d + 18a^2b^2 \\
& - 12a^2bc - 27a^2bd - 3a^2cd + 30a^2d^2 + 18ab^3 + 90ab^2c + 30ab^2d + 45abc^2 - 123abcd - 24abd^2 \\
& - 33ac^2d + 15acd^2 + 12ad^3 - 15b^3c - 39b^2c^2 - 15bc^3 + 63bc^2d - 18bcd^2 + 12c^3d - 27c^2d^2 \\
& - 3cd^3 + 9a^2b - 6a^2d - 27ab^2 - 42abc + 6abd + 27acd - 3ad^2 + 15b^2c + 15bc^2 - 3bcd - 3c^2d \\
& + 12cd^2 + 9ab - 3ad - 3bc - 3cd) s^2 - (abcd + 2abd^2 + acd^2 - b^2c^2 - 2b^2cd - bc^2d - 2abd \\
& - 2acd - 2ad^2 + b^2c + bc^2 + 3bcd + c^2d + 2ad - bc - cd)(2abcd + abd^2 - acd^2 - 2b^2c^2 - b^2cd
\end{aligned}$$

$$+ bc^2d - abd - acd - ad^2 + 2b^2c + 2bc^2 - c^2d + ad - 2bc + cd)(abcd - abd^2 - 2acd^2 - b^2c^2 + b^2cd + 2bc^2d + abd + acd + ad^2 + b^2c + bc^2 - 3bcd - 2c^2d - ad - bc + 2cd)).$$

6.8. **Class 2.4** :  $I_6^* + 4I_2 + 4I_1$  **with MWG** =  $\{0\}$ .

$$\begin{aligned} y^2 = & x^3 + s \left( s^4 - (3ac + ad + bc - 2a - 2b - 2c + d + 1)s^3 + (3a^2c^2 + 2a^2cd + 2abc^2 \right. \\ & + abcd - 4a^2c - a^2d - 3abc + abd - 4ac^2 - 2acd + ad^2 - b^2c - bc^2 - 2bcd + a^2 \\ & + 5ac - ad + b^2 + 3bc - bd + c^2 + cd - a - b - c + d)s^2 - (a^3c^3 + a^3c^2d + a^2bc^3 \\ & + a^2bc^2d - 2a^3c^2 - a^3cd - a^2bc^2 + 4a^2bcd - 2a^2c^3 - 3a^2c^2d - a^2cd^2 - 4ab^2c^2 \\ & - abc^3 - abc^2d + 2abcd^2 - 2b^2c^2d + a^3c + 4a^2c^2 - a^2cd - 2a^2d^2 + 4ab^2c + 5abc^2 \\ & - 4abcd + ac^3 + 2ac^2d - acd^2 + 2b^2c^2 + 2b^2cd + 2bc^2d - 2bcd^2 - 2a^2c + 2a^2d \\ & - 6abc - 2abd - 2ac^2 + 4acd + 2ad^2 - 2b^2c - 2bc^2 + 2cd^2 + 2ab + ac - 2ad \\ & \left. + 2bc - 2cd)s + 3ac(bc - c - d + 1)(a + d - 1)(ad - bc + b - d) \right) x^2 \\ & + s^2(s - ac - cd + c)(s + abc - ac - ad - bs + a) \left( (a - 1)(ad - bc)(c + d - 1)s^3 \right. \\ & - (ad - bc + b - d)(3a^2c^2 + 2a^2cd + 2abc^2 + abcd - 4a^2c - a^2d - 3abc - 4ac^2 \\ & - 2acd + ad^2 - bc^2 - 2bcd + a^2 + 5ac + 2bc + c^2 + cd - a - c)s^2 + (ad - bc + b \\ & - d)(2a^3c^3 + 2a^3c^2d + 2a^2bc^3 + 2a^2bc^2d - 4a^3c^2 - 2a^3cd - 2a^2bc^2 + 2a^2bcd \\ & - 4a^2c^3 - 6a^2c^2d - 2a^2cd^2 - 2ab^2c^2 - 2abc^3 - 2abc^2d + abcd^2 - b^2c^2d + 2a^3c \\ & + 8a^2c^2 + 4a^2cd - a^2d^2 + 2ab^2c + 4abc^2 - 2abcd + 2ac^3 + 4ac^2d + acd^2 + b^2c^2 \\ & + b^2cd + bc^2d - bcd^2 - 4a^2c + a^2d - 3abc - abd - 4ac^2 - acd + ad^2 - b^2c - bc^2 \\ & \left. + cd^2 + ab + 2ac - ad + bc - cd) - 3ac(bc - c - d + 1)(a + d - 1)(ad - bc + b - d) \right) x \\ & + ac(ad - bc + b - d)^2 s^3 (s - ac - cd + c)^2 ((1 - b)s + abc - ac - ad + a)^2 \left( (1 - a)(c + d \right. \\ & \left. - 1)(ac + bc - a - c - d + 1)s + (bc - c - d + 1)(a + d - 1)(ad - bc + b - d) \right). \end{aligned}$$

6.9. **Class 2.5** :  $I_4^* + 6I_2 + 2I_1$  **with MWG** =  $\mathbb{Z}/2\mathbb{Z}$ .

$$\begin{aligned} y^2 = & x^3 + t \left( ac(a - 1)(c + d - 1)(ac + bc - a - c - d + 1)t^3 + (4ac^2 - c^2 + a - 5ac \right. \\ & - cd - a^2 + 2acd + 2bcd - ad^2 - 3a^2c^2 + bc^2 - 2a^2cd + c + 3abc - abcd + a^2d \\ & - 2bc + 4a^2c - 2abc^2)t^2 + (ad + bc + 3ac - 2a - 2b - 2c + d + 1)t - 1 \Big) x^2 \\ & - t^4 \left( c(a + d - 1)t - 1 \right) \left( a(bc - c - d + 1)t - b + 1 \right) \left( (a - 1)(ad - bc)(c + d - 1)t \right. \\ & \left. - ad + bc - b + d \right) x. \end{aligned}$$

6.10. **Class 2.6** :  $I_4^* + I_0^* + 2I_2 + 4I_1$  **with MWG** =  $\{0\}$ .

$$\begin{aligned} y^2 = & x^3 - 27s^2 \left( s^4 + (ad + bc + a + b - 2c - 2d + 1)s^3 + (a^2d^2 - abcd + b^2c^2 + 4a^2d + 3abc \right. \\ & + 3abd - 2acd - ad^2 + 4b^2c - 6bc^2 - 7bcd + a^2 + 3ab - 6ac - 9ad + b^2 + bc - bd + 6c^2 \\ & \left. + 8cd + d^2 + 4a - b - 6c - d + 1)s^2 + (2a^3d^2 - 3a^2bcd + a^2bd^2 - a^2cd^2 + ab^2c^2 - 3ab^2cd \right. \end{aligned}$$



$$\begin{aligned}
& + 3abc^2d + 3abcd^2 + 2b^3c^2 - 2b^2c^3 - 3b^2c^2d + 2a^3d + a^2bc + 4a^2bd - a^2cd - 2a^2d^2 \\
& + 4ab^2c + ab^2d - 6abc^2 - 12abcd - 2abd^2 - 2ac^2d - acd^2 + 2b^3c - b^2cd + 6bc^3 \\
& + 7bc^2d + a^2b - 2a^2c - 6a^2d + ab^2 + 5abc + abd + 6ac^2 + 12acd + 2ad^2 - 4b^2c \\
& - 8bc^2 - 4c^3 - 4c^2d + 2a^2 - 3ab - 8ac - 2ad + 4bc + 6c^2 + cd + 2a - 2c)s \\
& + a^4d^2 - 2a^3bcd + a^3bd^2 - a^3cd^2 + a^2b^2c^2 - 2a^2b^2cd + a^2b^2d^2 + 2a^2bc^2d \\
& + a^2bcd^2 + a^2c^2d^2 + ab^3c^2 - 2ab^3cd - ab^2c^3 - 2ab^2c^2d - 2abc^3d + b^4c^2 \\
& + b^3c^3 + b^2c^4 + a^3bd + a^3cd - a^3d^2 - a^2b^2c - a^2b^2d - a^2bc^2 - 2a^2bcd - 2a^2bd^2 \\
& - 3a^2c^2d - a^2cd^2 + ab^3c + 3ab^2c^2 + 5ab^2cd + 3abc^3 + 5abc^2d + 2ac^3d - 3b^3c^2 \\
& - 4b^2c^3 - 2bc^4 - a^3d + a^2b^2 + 3a^2bc + 2a^2bd + a^2c^2 + 4a^2cd + a^2d^2 - 4ab^2c \\
& - 8abc^2 - 4abcd - 2ac^3 - 3ac^2d + 4b^2c^2 + 5bc^3 + c^4 - 2a^2b - 2a^2c - a^2d \\
& + 5abc + 4ac^2 + acd - 3bc^2 - 2c^3 + 2 + a^2 - 2ac + c^2)x \\
& - 27s^3 \left( 2s^6 + 6(ad + bc + a + b - 2c - 2d + 1)s^5 + 3(2a^2d^2 + abcd + 2b^2c^2 + 6a^2d \right. \\
& + 5abc + 5abd - 6acd - 5ad^2 + 6b^2c - 10bc^2 - 11bcd + 2a^2 + 5ab - 10ac - 11ad \\
& + 2b^2 - bc - 5bd + 10c^2 + 16cd + 5d^2 + 6a + b - 10c - 5d + 2)s^4 + (2a^3d^3 \\
& - 3a^2bcd^2 - 3ab^2c^2d + 2b^3c^3 + 18a^3d^2 + 3a^2bcd + 12a^2bd^2 - 9a^2cd^2 - 3a^2d^3 \\
& + 12ab^2c^2 + 3ab^2cd + 3abc^2d + 3abcd^2 + 18b^3c^2 - 24b^2c^3 - 30b^2c^2d + 18a^3d \\
& + 12a^2bc + 36a^2bd - 39a^2cd - 48a^2d^2 + 36ab^2c + 12ab^2d - 60abc^2 - 108abcd \\
& - 18abd^2 + 12ac^2d + 15acd^2 - 3ad^3 + 18b^3c - 27b^2c^2 - 39b^2cd + 60bc^3 + 99bc^2d \\
& + 33bcd^2 + 2a^3 + 12a^2b - 24a^2c - 27a^2d + 12ab^2 + 9abc - 33abd + 60ac^2 \\
& + 114acd + 57ad^2 + 2b^3 - 6b^2c - 3b^2d - 48bc^2 - 12bcd - 3bd^2 - 40c^3 - 72c^2d \\
& - 30cd^2 + 2d^3 + 18a^2 + 3ab - 72ac - 48ad - 3b^2 + 15bc + 12bd + 60c^2 + 45cd \\
& - 3d^2 + 18a - 3b - 24c - 3d + 2)s^3 + 3(2a^4d^3 - 4a^3bcd^2 + a^3bd^3 - a^3cd^3 \\
& + a^2b^2c^2d + a^2b^2cd^2 - a^2bc^2d^2 - 6a^2bcd^3 + ab^3c^3 - 4ab^3c^2d + 4ab^2c^3d + 9ab^2c^2d^2 \\
& + 2b^4c^3 - 2b^3c^4 - 3b^3c^3d + 6a^4d^2 - a^3bcd + 9a^3bd^2 - 5a^3cd^2 - 3a^3d^3 + a^2b^2c^2 \\
& - 8a^2b^2cd + a^2b^2d^2 + 6a^2bc^2d + 12a^2bcd^2 + 2a^2bd^3 + 2a^2c^2d^2 + 4a^2cd^3 + 9ab^3c^2 \\
& - ab^3cd - 12ab^2c^3 - 24ab^2c^2d - 6ab^2cd^2 - 9abc^3d - 12abc^2d^2 + 3abcd^3 + 6b^4c^2 \\
& - 7b^3c^3 - 11b^3c^2d + 12b^2c^4 + 20b^2c^3d + 6b^2c^2d^2 + 2a^4d + a^3bc + 9a^3bd - 8a^3cd \\
& - 18a^3d^2 + 10a^2b^2c + 10a^2b^2d - 12a^2bc^2 - 37a^2bcd - 20a^2bd^2 + 3a^2c^2d + 5a^2cd^2 \\
& - 3a^2d^3 + 9ab^3c + ab^3d - 2ab^2c^2 - 7ab^2cd + 2ab^2d^2 + 30abc^3 + 70abc^2d + 26abcd^2 \\
& - 2abd^3 + 4ac^3d + 5ac^2d^2 - acd^3 + 2b^4c - 6b^3c^2 - 2b^3cd - 9b^2c^3 + 3b^2c^2d - b^2cd^2 \\
& - 20bc^4 - 33bc^3d - 11bc^2d^2 + a^3b - 2a^3c - a^3d + a^2b^2 + 7a^2bc - 5a^2bd + 12a^2c^2 \\
& + 30a^2cd + 27a^2d^2 + ab^3 - 11ab^2c - 9ab^2d - 39abc^2 - 24abcd + abd^2 - 20ac^3 \\
& - 48ac^2d - 25acd^2 + 2ad^3 - 5b^3c + 9b^2c^2 + 10b^2cd + 34bc^3 + 23bc^2d + 10c^4 \\
& \left. + 16c^3d + 5c^2d^2 + 2a^3 - a^2b - 18a^2c - 18a^2d + ab^2 + 19abc + 11abd + 36ac^2 \right)
\end{aligned}$$

$$\begin{aligned}
& + 36acd - 3ad^2 - 21bc^2 - 10bcd - 20c^3 - 15c^2d + cd^2 + 6a^2 - 4ab - 18ac \\
& - 3ad + 5bc + 12c^2 + 2cd + 2a - 2c)s^2 + 3(2a^5d^3 - 5a^4bcd^2 + 2a^4bd^3 - 2a^4cd^3 \\
& + 4a^3b^2c^2d - 2a^3b^2cd^2 - a^3b^2d^3 + 2a^3bc^2d^2 - 10a^3bcd^3 - a^3c^2d^3 - a^2b^3c^3 \\
& - 2a^2b^3c^2d + 4a^2b^3cd^2 + 2a^2b^2c^3d + 19a^2b^2c^2d^2 - 3a^2b^2cd^3 + 4a^2bc^3d^2 + 3a^2bc^2d^3 \\
& + 2ab^4c^3 - 5ab^4c^2d - 2ab^3c^4 - 8ab^3c^3d + 6ab^3c^2d^2 - 5ab^2c^4d - 6ab^2c^3d^2 + 2b^5c^3 \\
& - b^4c^4 - 3b^4c^3d + 2b^3c^5 + 3b^3c^4d + 2a^5d^2 - a^4bcd + 6a^4bd^2 - a^4cd^2 - 3a^4d^3 - a^3b^2c^2 \\
& - 11a^3b^2cd + 5a^3b^2d^2 + 3a^3bc^2d + 19a^3bcd^2 + 4a^3bd^3 + 3a^3c^2d^2 + 8a^3cd^3 + 5a^2b^3c^2 \\
& - 11a^2b^3cd - a^2b^3d^2 - 2a^2b^2c^3 - 21a^2b^2c^2d - 12a^2b^2cd^2 + 2a^2b^2d^3 - 15a^2bc^3d \\
& - 38a^2bc^2d^2 + 5a^2bcd^3 - 3a^2c^3d^2 - a^2c^2d^3 + 6ab^4c^2 - ab^4cd + 5ab^3c^3 + 10ab^3c^2d \\
& - ab^3cd^2 + 12ab^2c^4 + 34ab^2c^3d - 7ab^2c^2d^2 + 11abc^4d + 11abc^3d^2 + 2b^5c^2 - 2b^4c^3 \\
& - b^4c^2d - 4b^3c^4 + 2b^3c^3d - 8b^2c^5 - 10b^2c^4d + 2a^4bd - a^4cd - 8a^4d^2 + 4a^3b^2c \\
& + 5a^3b^2d - 2a^3bc^2 - 12a^3bcd - 22a^3bd^2 - 2a^3c^2d - 5a^3cd^2 - 3a^3d^3 + 5a^2b^3c \\
& + 4a^2b^3d + 4a^2b^2c^2 + 24a^2b^2cd - a^2b^2d^2 + 12a^2bc^3 + 57a^2bc^2d + 28a^2bcd^2 - 4a^2bd^3 \\
& + 9a^2c^3d + 15a^2c^2d^2 - 2a^2cd^3 + 2ab^4c - 19ab^3c^2 - 4ab^3cd - 32ab^2c^3 - 26ab^2c^2d \\
& + 4ab^2cd^2 - 20abc^4 - 52abc^3d - 5abc^2d^2 - 6ac^4d - 5ac^3d^2 - 7b^4c^2 + 5b^3c^3 \\
& + 6b^3c^2d + 21b^2c^4 + 10b^2c^3d + 10bc^5 + 11bc^4d + a^4d - a^3b^2 + 3a^3bc + 3a^3bd \\
& + 2a^3c^2 + 10a^3cd + 17a^3d^2 - a^2b^3 - 12a^2b^2c - 11a^2b^2d - 26a^2bc^2 - 40a^2bcd \\
& + 5a^2bd^2 - 8a^2c^3 - 33a^2c^2d - 20a^2cd^2 + 2a^2d^3 + 36ab^2c^2 + 15ab^2cd + 49abc^3 \\
& + 45abc^2d - 5abcd^2 + 10ac^4 + 26ac^3d + 6ac^2d^2 + 5b^3c^2 - 20b^2c^3 - 10b^2c^2d \\
& - 26bc^4 - 14bc^3d - 4c^5 - 4c^4d - a^3b - 4a^3c - 8a^3d + 4a^2b^2 + 19a^2bc \\
& + 10a^2bd + 18a^2c^2 + 27a^2cd - 3a^2d^2 - 10ab^2c - 41abc^2 - 14abcd - 24ac^3 \\
& - 24ac^2d + 2acd^2 + 5b^2c^2 + 23bc^3 + 6bc^2d + 10c^4 + 5c^3d + 2a^3 - 5a^2b \\
& - 12a^2c - 3a^2d + 12abc + 18ac^2 + 4acd - 7bc^2 - 8c^3 - c^2d + 2a^2 - 4ac + 2c^2)s \\
& + (a^2d - abc + 2abd + acd - 2b^2c - bc^2 - ab - ac - 2ad + 3bc + c^2 + a - c)(2a^2d \\
& - 2abc + abd - acd - b^2c + bc^2 + ab + ac - ad - c^2 - a + c)(a^2d - abc - abd \\
& - 2acd + b^2c + 2bc^2 + 2ab + 2ac + ad - 3bc - 2c^2 - 2a + 2c).
\end{aligned}$$

6.11. **Class 2.7** :  $I_2^* + 8I_2$  with  $\text{MWG} = (\mathbb{Z}/2\mathbb{Z})^2$ .

$$\begin{aligned}
y^2 = & \left( x - t(bt + ct - t - 1)(abt - bct - a + 1) \right) \\
& \times \left( x - at(bt - dt - 1)(bt + ct - t - 1) \right) \\
& \times \left( x - t(1 - ct)(bt - dt - 1)(abt - bct - a + 1) \right).
\end{aligned}$$

6.12. **Class 2.8** :  $I_2^* + I_0^* + 4I_2$  with  $\text{MWG} = \mathbb{Z}/2\mathbb{Z}$ .

$$\begin{aligned}
y^2 = & x^3 + t \left( (ad + bc + acd - bc^2 - 2cd)t^2 + (ad - 2bc - 2a + b + c - 2d + 1)t - b + 1 \right) x^2 \\
& + t^3(cdt + d - 1)((a - c)t - 1)((ad - bc + b - d)t - a - b + 1)x.
\end{aligned}$$

6.13. **Class 2.9** :  $2\mathbf{I}_2^* + 2\mathbf{I}_2 + 4\mathbf{I}_1$  **with MWG** =  $\{0\}$ .

$$\begin{aligned}
y^2 = & x^3 - 27t^2 \left( (c+d-1)^2(a-1)^2t^4 - (a^2cd - 2a^2d^2 - 2abc^2 + abcd + a^2c + 3a^2d \right. \\
& + 3abc + abd + ac^2 + acd + 6ad^2 - bc^2 - 4bcd - a^2 - ab - 4ac - 9ad + 3bc \\
& + 2bd - c^2 - 2cd - 4d^2 + 3a - 2b + 3c + 6d - 2)t^3 + (a^2d^2 - abcd + b^2c^2 - a^2d \\
& - 2abc - 2abd + 3acd - 6ad^2 - b^2c - bc^2 + 8bcd + a^2 + 3ab - ac + 6ad + b^2 \\
& - 4bc - 6bd + c^2 - 2cd + 6d^2 - a + 4b - c - 6d + 1)t^2 - (abd + 2ad^2 + b^2c \\
& - 4bcd - 2ab - ad - 2b^2 + bc + 6bd + 2cd - 4d^2 - 2b + 2d)t + (b-d)^2 \Big) x \\
& + 27t^3 \left( 2(c+d-1)^3(a-1)^3t^6 - 3(a-1)(c+d-1)(a^2cd - 2a^2d^2 - 2abc^2 \right. \\
& + abcd + a^2c + 3a^2d + 3abc + abd + ac^2 + acd + 6ad^2 - bc^2 - 4bcd - a^2 \\
& - ab - 4ac - 9ad + 3bc + 2bd - c^2 - 2cd - 4d^2 + 3a - 2b + 3c + 6d - 2)t^5 \\
& - 3(a^3cd^2 - 2a^3d^3 + 2a^2bc^2d + 2a^2bcd^2 - 2ab^2c^3 + ab^2c^2d - 4a^3cd + 4a^3d^2 \\
& + 3a^2bc^2 - 5a^2bcd + 3a^2bd^2 + a^2c^2d - 2a^2cd^2 + 12a^2d^3 + 4ab^2c^2 - 4ab^2cd \\
& + 2abc^3 + 2abc^2d - 15abcd^2 - b^2c^3 + 5b^2c^2d + a^3c - a^3d - 2a^2bc - 2a^2bd \\
& - 4a^2c^2 + a^2cd - 24a^2d^2 - ab^2c + ab^2d - 11abc^2 + 13abcd + 4abd^2 + ac^3 \\
& - 3ac^2d - 3acd^2 - 20ad^3 - b^2c^2 - 5b^2cd + 4bc^3 + 5bc^2d + 16bcd^2 - a^3 \\
& - a^2b + 5a^2c + 13a^2d - ab^2 + 8abc - 5abd + 5ac^2 + 12acd + 40ad^2 + 4b^2c \\
& + 2b^2d - 10bc^2 - 26bcd - 10bd^2 - c^3 + 2c^2d + 4cd^2 + 10d^3 - a^2 + ab - 10ac \\
& - 24ad - 2b^2 + 12bc + 16bd - c^2 - 9cd - 20d^2 + 4a - 6b + 4c + 12d - 2)t^4 \\
& + (2a^3d^3 - 3a^2bcd^2 - 3ab^2c^2d + 2b^3c^3 - 3a^3d^2 + 24a^2bcd - 9a^2bd^2 + 12a^2cd^2 \\
& - 24a^2d^3 - 9ab^2c^2 + 24ab^2cd - 18abc^2d + 45abcd^2 - 3b^3c^2 - 3b^2c^3 - 30b^2c^2d \\
& - 3a^3d - 9a^2bc - 6a^2bd - 18a^2cd + 36a^2d^2 - 6ab^2c - 9ab^2d + 24abc^2 - 24abcd \\
& - 18abd^2 + 12ac^2d - 27acd^2 + 60ad^3 - 3b^3c + 15b^2c^2 + 45b^2cd - 3bc^3 + 15bc^2d \\
& - 72bcd^2 + 2a^3 + 12a^2b - 3a^2c - 6a^2d + 12ab^2 - 33abc + 9abd - 3ac^2 + 9acd \\
& - 90ad^2 + 2b^3 - 27b^2c - 24b^2d + 15bc^2 + 72bcd + 60bd^2 + 2c^3 - 9c^2d + 12cd^2 \\
& - 40d^3 - 3a^2 + 3ab + 12ac + 36ad + 18b^2 - 27bc - 72bd - 3c^2 + 3cd + 60d^2 \\
& - 3a + 18b - 3c - 24d + 2)t^3 - 3(a^2bd^2 + 2a^2d^3 - 4ab^2cd - 5abcd^2 + b^3c^2 \\
& + 5b^2c^2d + 2a^2bd - 2a^2d^2 + 3ab^2c + 3ab^2d + abcd + 4abd^2 + 5acd^2 - 10ad^3 \\
& + 2b^3c - 4b^2c^2 - 15b^2cd - 5bc^2d + 16bcd^2 - 2a^2b - a^2d - 5ab^2 + 2abc + abd \\
& - 2acd + 10ad^2 - 2b^3 + 6b^2c + 12b^2d + bc^2 - 6bcd - 20bd^2 + 2c^2d - 6cd^2 + 10d^3 \\
& - ab - 2ad - 6b^2 + 2bc + 16bd + cd - 10d^2 - 2b + 2d)t^2 - 3(b-d)(abd + 2ad^2 \\
& + b^2c - 4bcd - 2ab - ad - 2b^2 + bc + 6bd + 2cd - 4d^2 - 2b + 2d)t + 2(b-d)^3 \Big).
\end{aligned}$$

6.14. **Class 2.10** :  $\mathbf{I}_2^* + 2\mathbf{I}_0^* + 8\mathbf{I}_1$  **with MWG** =  $\{0\}$ .

$$y^2 = x^3 - 27(ad - bc)^2(s - ab + ad)(s - ab + bc) \left( (a^2d^2 - 2abcd + b^2c^2 + abd + acd - b^2c \right.$$

$$\begin{aligned}
& -bc^2 - ad + b^2 + 3bc + c^2 - 2b - 2c + 1)s^2 - (2a^3bd^2 - 2a^3d^3 - 4a^2b^2cd \\
& + 2a^2bcd^2 + 2ab^3c^2 + 2ab^2c^2d - 2b^3c^3 + 2a^3d^2 + 2a^2b^2d + a^2bcd - a^2bd^2 - a^2cd^2 \\
& + 2a^2d^3 - 2ab^3c - 3ab^2c^2 - ab^2cd - abc^2d - abcd^2 + 2b^3c^2 + 2b^2c^3 - b^2c^2d - a^2bd \\
& - 2a^2cd - 4a^2d^2 + 2ab^3 + 5ab^2c + 4abc^2 + 2abcd - 2abd^2 + acd^2 - 2b^3c - 2b^2c^2 \\
& + 2b^2cd - 2bc^3 - bc^2d + 2a^2d - 4ab^2 - 6abc + 2abd + 2acd + 2ad^2 + 2b^2c + 2bc^2 \\
& - 2bcd + 2ab - 2ad)s + a^4b^2d^2 - 2a^4bd^3 + a^4d^4 - 2a^3b^3cd + 2a^3b^2cd^2 + a^2b^4c^2 \\
& + 2a^2b^3c^2d - 2a^2b^2c^2d^2 - 2ab^4c^3 + b^4c^4 + 2a^4bd^2 - 2a^4d^3 + a^3b^3d - a^3b^2d^2 \\
& + a^3bcd^2 + 2a^3bd^3 - 2a^3d^4 - a^2b^4c - 2a^2b^3c^2 - a^2b^3cd - 2a^2b^2c^2d + a^2bc^2d^2 \\
& + 2a^2bcd^3 + 2ab^4c^2 + 3ab^3c^3 - ab^3c^2d - ab^2c^2d^2 - b^4c^3 - b^3c^4 + b^3c^3d + a^4d^2 \\
& - 4a^3bcd - 4a^3bd^2 + 4a^3d^3 + a^2b^4 + 2a^2b^3c + 4a^2b^2c^2 + a^2b^2cd - 2a^2b^2d^2 \\
& + 2a^2bc^2d - 3a^2bcd^2 + a^2d^4 - 2ab^4c - ab^3c^2 + 2ab^3cd - 4ab^2c^3 + 2ab^2c^2d \\
& + 2ab^2cd^2 - abc^2d^2 - 2abcd^3 + b^4c^2 - b^3c^3 - 2b^3c^2d + b^2c^4 + b^2c^3d + b^2c^2d^2 \\
& + 2a^3bd - 2a^3d^2 - 2a^2b^3 - 4a^2b^2c + 2a^2b^2d + 4a^2bcd + 2a^2bd^2 - 2a^2d^3 \\
& + 2ab^3c + 2ab^2c^2 - 4ab^2cd - 2abc^2d + 2abcd^2 + a^2b^2 - 2a^2bd + a^2d^2) x \\
& - 27(ad - bc)^3(s - ab + ad)^3(s - ab + bc)^3 \Big( (2ad - 2bc + b + c - 1)(ad - bc + \\
& 2b + 2c - 2)(ad - bc - b - c + 1)s^3 - 3(2a^4bd^3 - 2a^4d^4 - 6a^3b^2cd^2 + 4a^3bcd^3 \\
& + 6a^2b^3c^2d - 2ab^4c^3 - 4ab^3c^3d + 2b^4c^4 + 2a^4d^3 + 3a^3b^2d^2 - 2a^3bd^3 - 2a^3cd^3 \\
& + 2a^3d^4 - 6a^2b^3cd - 6a^2b^2c^2d + a^2b^2cd^2 + a^2bc^2d^2 - 3a^2bcd^3 + 3ab^4c^2 + 4ab^3c^3 \\
& + 4ab^3c^2d + 4ab^2c^3d - 3b^4c^3 - 3b^3c^4 + b^3c^3d - a^3bd^2 - a^3cd^2 - 3a^3d^3 - 3a^2b^3d \\
& + 2a^2b^2cd + a^2b^2d^2 - 4a^2bc^2d - 2a^2bcd^2 - a^2bd^3 + a^2c^2d^2 + 2a^2cd^3 + 3ab^4c - ab^3c^2 \\
& + 2ab^3cd + 5ab^2c^3 + 3ab^2c^2d - ab^2cd^2 + 2abc^3d + 2abc^2d^2 - 3b^4c^2 + 2b^3c^3 + 2b^3c^2d \\
& - 3b^2c^4 - 4b^2c^3d + a^3d^2 + 5a^2b^2d + 8a^2bcd + 2a^2bd^2 + 2a^2c^2d + 2a^2cd^2 + a^2d^3 \\
& - 2ab^4 - 11ab^3c - 15ab^2c^2 - 3ab^2cd + 2ab^2d^2 - 4abc^3 - 3abc^2d + 2abcd^2 - ac^2d^2 \\
& + 2b^4c + 7b^3c^2 - 2b^3cd + 7b^2c^3 - 3b^2c^2d + 2bc^4 + bc^3d - 4a^2bd - 4a^2cd - 3a^2d^2 \\
& + 6ab^3 + 16ab^2c - 2ab^2d + 10abc^2 - 3abcd - 4abd^2 - 2ac^2d - acd^2 - 4b^3c \\
& - 6b^2c^2 + 4b^2cd - 4bc^3 + bc^2d + 2a^2d - 6ab^2 - 8abc + 4abd + 4acd + 2ad^2 + 2b^2c \\
& + 2bc^2 - 2bcd + 2ab - 2ad)s^2 + 3(2a^5b^2d^3 - 4a^5bd^4 + 2a^5d^5 - 6a^4b^3cd^2 + 8a^4b^2cd^3 \\
& - 2a^4bcd^4 + 6a^3b^4c^2d - 4a^3b^2c^2d^3 - 2a^2b^5c^3 - 8a^2b^4c^3d + 4a^2b^3c^3d^2 + 4ab^5c^4 \\
& + 2ab^4c^4d - 2b^5c^5 + 4a^5bd^3 - 4a^5d^4 + 3a^4b^3d^2 - 3a^4b^2cd^2 - 4a^4b^2d^3 + a^4bcd^3 \\
& + 5a^4bd^4 + a^4cd^4 - 4a^4d^5 - 6a^3b^4cd - 6a^3b^3c^2d + 2a^3b^3cd^2 + 2a^3b^2c^2d^2 \\
& - 4a^3b^2cd^3 + 2a^3bc^2d^3 + 5a^3bcd^4 + 3a^2b^5c^2 + 5a^2b^4c^3 + 8a^2b^4c^2d + 9a^2b^3c^3d \\
& - 4a^2b^3c^2d^2 - 4a^2b^2c^3d^2 - 6ab^5c^3 - 8ab^4c^4 - 2ab^3c^4d + ab^3c^3d^2 + 3b^5c^4 + 3b^4c^5 \\
& - 2b^4c^4d + 2a^5d^3 + a^4b^2d^2 - 5a^4bcd^2 - 8a^4bd^3 + a^4cd^3 + 10a^4d^4 - 3a^3b^4d \\
& + 4a^3b^3cd + 2a^3b^3d^2 - 5a^3b^2c^2d - 4a^3b^2cd^2 - 2a^3b^2d^3 + 2a^3bc^2d^2 - 6a^3bcd^3
\end{aligned}$$

$$\begin{aligned}
& + a^3bd^4 - 2a^3cd^4 + 2a^3d^5 + 3a^2b^5c - 5a^2b^4c^2 + 4a^2b^4cd + 8a^2b^3c^3 - 4a^2b^3cd^2 \\
& + 7a^2b^2c^3d + 13a^2b^2c^2d^2 - 2a^2bc^3d^2 - 3a^2bcd^4 - 6ab^5c^2 + 12ab^4c^3 + 3ab^4c^3d \\
& - 10ab^3c^4 - 10ab^3c^3d + 3ab^3c^2d^2 - ab^2c^4d - 6ab^2c^3d^2 + 3b^5c^3 - 7b^4c^4 \\
& - 4b^4c^3d + 3b^3c^5 + 8b^3c^4d + b^3c^3d^2 + 3a^4bd^2 - 2a^4cd^2 - 8a^4d^3 + 4a^3b^3d \\
& + 8a^3b^2cd + 4a^3b^2d^2 + 8a^3bc^2d + 13a^3bcd^2 - 2a^3cd^3 - 8a^3d^4 - 2a^2b^5 - 10a^2b^4c \\
& - 17a^2b^3c^2 - 4a^2b^3cd + 4a^2b^3d^2 - 8a^2b^2c^3 - 8a^2b^2c^2d + 4a^2b^2cd^2 - 4a^2bc^3d \\
& - 2a^2bc^2d^2 + 9a^2bcd^3 - 2a^2bd^4 + a^2cd^4 + 4ab^5c + 12ab^4c^2 - 4ab^4cd + 15ab^3c^3 \\
& - 8ab^3c^2d - 4ab^3cd^2 + 8ab^2c^4 - 2ab^2c^2d^2 + 4ab^2cd^3 + 2abc^3d^2 - 2abc^2d^3 \\
& - 2b^5c^2 - 2b^4c^3 + 4b^4c^2d - 2b^3c^4 + b^3c^3d - 2b^3c^2d^2 - 2b^2c^5 - 2b^2c^4d \\
& + b^2c^3d^2 + 2a^4d^2 - 5a^3b^2d - 12a^3bcd - 5a^3bd^2 + 4a^3cd^2 + 10a^3d^3 + 6a^2b^4 \\
& + 17a^2b^3c - 4a^2b^3d + 16a^2b^2c^2 - 6a^2b^2cd - 8a^2b^2d^2 - 4a^2bc^2d - 12a^2bcd^2 \\
& + 4a^2bd^3 + a^2cd^3 + 2a^2d^4 - 8ab^4c - 13ab^3c^2 + 12ab^3cd - 12ab^2c^3 + 13ab^2c^2d \\
& + 4abc^3d - 4abcd^3 + 2b^4c^2 + b^3c^3 - 4b^3c^2d + 2b^2c^4 - b^2c^3d + 2b^2c^2d^2 + 4a^3bd \\
& - 4a^3d^2 - 6a^2b^3 - 10a^2b^2c + 8a^2b^2d + 12a^2bcd + 2a^2bd^2 - 2a^2cd^2 - 4a^2d^3 \\
& + 4ab^3c + 4ab^2c^2 - 8ab^2cd - 4abc^2d + 4abcd^2 + 2a^2b^2 - 4a^2bd + 2a^2d^2)s \\
& - 2a^6b^3d^3 + 6a^6b^2d^4 - 6a^6bd^5 + 2a^6d^6 + 6a^5b^4cd^2 - 12a^5b^3cd^3 + 6a^5b^2cd^4 \\
& - 6a^4b^5c^2d + 12a^4b^3c^2d^3 - 6a^4b^2c^2d^4 + 2a^3b^6c^3 + 12a^3b^5c^3d - 12a^3b^4c^3d^2 \\
& - 6a^2b^6c^4 - 6a^2b^5c^4d + 6a^2b^4c^4d^2 + 6ab^6c^5 - 2b^6c^6 - 6a^6b^2d^3 + 12a^6bd^4 \\
& - 6a^6d^5 - 3a^5b^4d^2 + 6a^5b^3cd^2 + 6a^5b^3d^3 - 9a^5b^2cd^3 - 9a^5b^2d^4 + 3a^5bcd^4 \\
& + 12a^5bd^5 - 6a^5d^6 + 6a^4b^5cd + 6a^4b^4c^2d - 3a^4b^4cd^2 - 3a^4b^3c^2d^2 + 3a^4b^3cd^3 \\
& - 3a^4b^2c^2d^3 - 12a^4b^2cd^4 + 3a^4bc^2d^4 + 6a^4bcd^5 - 3a^3b^6c^2 - 6a^3b^5c^3 \\
& - 12a^3b^5c^2d - 15a^3b^4c^3d + 12a^3b^4c^2d^2 + 9a^3b^3c^3d^2 + 3a^3b^2c^2d^4 + 9a^2b^6c^3 \\
& + 15a^2b^5c^4 + 3a^2b^5c^3d + 9a^2b^4c^4d - 9a^2b^4c^3d^2 - 6a^2b^3c^4d^2 - 3a^2b^3c^3d^3 \\
& - 9ab^6c^4 - 12ab^5c^5 + 6ab^5c^4d + 3ab^4c^4d^2 + 3b^6c^5 + 3b^5c^6 - 3b^5c^5d \\
& - 6a^6bd^3 + 6a^6d^4 - 3a^5b^3d^2 + 12a^5b^2cd^2 + 15a^5b^2d^3 - 15a^5bcd^3 - 30a^5bd^4 \\
& + 18a^5d^5 + 3a^4b^5d - 6a^4b^4cd - 3a^4b^4d^2 + 6a^4b^3c^2d + 6a^4b^3cd^2 + 3a^4b^3d^3 \\
& + 6a^4b^2c^2d^2 + 18a^4b^2cd^3 - 3a^4b^2d^4 + 3a^4bc^2d^3 - 18a^4bcd^4 - 6a^4bd^5 + 6a^4d^6 \\
& - 3a^3b^6c + 9a^3b^5c^2 - 6a^3b^5cd - 12a^3b^4c^3 + 9a^3b^4c^2d + 9a^3b^4cd^2 - 21a^3b^3c^3d \\
& - 27a^3b^3c^2d^2 + 3a^3b^2c^3d^2 + 15a^3b^2c^2d^3 + 12a^3b^2cd^4 - 6a^3bc^2d^4 - 12a^3bcd^5 \\
& + 9a^2b^6c^2 - 30a^2b^5c^3 - 3a^2b^5c^2d + 24a^2b^4c^4 + 6a^2b^4c^3d - 12a^2b^4c^2d^2 \\
& + 9a^2b^3c^4d + 18a^2b^3c^3d^2 - 3a^2b^3c^2d^3 - 3a^2b^2c^4d^2 + 6a^2b^2c^3d^3 + 9a^2b^2c^2d^4 \\
& - 9ab^6c^3 + 33ab^5c^4 + 12ab^5c^3d - 15ab^4c^5 - 21ab^4c^4d + 3ab^4c^3d^2 - 12ab^3c^4d^2 \\
& - 6ab^3c^3d^3 + 3b^6c^4 - 12b^5c^5 - 6b^5c^4d + 3b^4c^6 + 12b^4c^5d + 3b^4c^4d^2 - 2a^6d^3 \\
& - 6a^5b^2d^2 + 12a^5bcd^2 + 24a^5bd^3 - 18a^5d^4 - 3a^4b^4d - 6a^4b^3cd - 6a^4b^3d^2 \\
& - 24a^4b^2c^2d - 30a^4b^2cd^2 + 3a^4b^2d^3 - 6a^4bc^2d^2 + 36a^4bcd^3 + 24a^4bd^4
\end{aligned}$$

$$\begin{aligned}
& -18a^4d^5 + 2a^3b^6 + 9a^3b^5c + 18a^3b^4c^2 + 3a^3b^4cd - 6a^3b^4d^2 + 16a^3b^3c^3 \\
& + 9a^3b^3c^2d - 6a^3b^3cd^2 + 24a^3b^2c^3d - 21a^3b^2c^2d^2 - 33a^3b^2cd^3 + 6a^3b^2d^4 \\
& - 6a^3bc^2d^3 + 27a^3bcd^4 - 2a^3d^6 - 6a^2b^6c - 15a^2b^5c^2 + 6a^2b^5cd - 21a^2b^4c^3 \\
& + 18a^2b^4c^2d + 12a^2b^4cd^2 - 24a^2b^3c^4 + 21a^2b^3c^3d + 12a^2b^3c^2d^2 - 12a^2b^3cd^3 \\
& - 6a^2b^2c^4d - 18a^2b^2c^2d^3 - 6a^2b^2cd^4 + 3a^2bc^2d^4 + 6a^2bcd^5 + 6ab^6c^2 + 3ab^5c^3 \\
& - 12ab^5c^2d - 12ab^4c^3d + 12ab^3c^5 - 3ab^3c^4d + 15ab^3c^3d^2 + 12ab^3c^2d^3 + 3ab^2c^4d^2 \\
& - 6ab^2c^3d^3 - 6ab^2c^2d^4 - 2b^6c^3 + 3b^5c^4 + 6b^5c^3d + 3b^4c^5 - 6b^4c^4d - 6b^4c^3d^2 \\
& - 2b^3c^6 - 3b^3c^5d + 3b^3c^4d^2 + 2b^3c^3d^3 - 6a^5bd^2 + 6a^5d^3 + 6a^4b^3d + 24a^4b^2cd \\
& + 6a^4b^2d^2 - 24a^4bcd^2 - 30a^4bd^3 + 18a^4d^4 - 6a^3b^5 - 18a^3b^4c + 6a^3b^4d \\
& - 24a^3b^3c^2 + 9a^3b^3cd + 12a^3b^3d^2 + 12a^3b^2c^2d + 36a^3b^2cd^2 - 12a^3b^2d^3 \\
& + 12a^3bc^2d^2 - 27a^3bcd^3 - 6a^3bd^4 + 6a^3d^5 + 12a^2b^5c + 21a^2b^4c^2 - 24a^2b^4cd \\
& + 24a^2b^3c^3 - 39a^2b^3c^2d - 24a^2b^2c^3d + 15a^2b^2c^2d^2 + 24a^2b^2cd^3 + 3a^2bc^2d^3 \\
& - 12a^2bcd^4 - 6ab^5c^2 - 3ab^4c^3 + 18ab^4c^2d - 6ab^3c^4 + 6ab^3c^3d - 18ab^3c^2d^2 \\
& + 6ab^2c^4d - 3ab^2c^3d^2 + 6ab^2c^2d^3 - 6a^4b^2d + 12a^4bd^2 - 6a^4d^3 + 6a^3b^4 \\
& + 12a^3b^3c - 12a^3b^3d - 24a^3b^2cd + 12a^3bcd^2 + 12a^3bd^3 - 6a^3d^4 - 6a^2b^4c \\
& - 6a^2b^3c^2 + 18a^2b^3cd + 12a^2b^2c^2d - 18a^2b^2cd^2 - 6a^2bc^2d^2 + 6a^2bcd^3 - 2a^3b^3 \\
& + 6a^3b^2d - 6a^3bd^2 + 2a^3d^3 \Big).
\end{aligned}$$

6.15. **Class 2.11** :  $2I_2^* + 6I_2$  **with MWG** =  $\mathbb{Z}/2\mathbb{Z}$ .

$$y^2 = \left(x - dt(t-1)(at-1)\right) \left(x - bt(t-1)(ct-1)\right) \left(x - t(t-1)(at-1)(ct-1)\right).$$

6.16. **Class 2.12** :  $3I_0^* + 2I_2 + 2I_1$  **with MWG** =  $\mathbb{Z}/2\mathbb{Z}$ .

$$\begin{aligned}
y^2 &= x^3 + 9t(t-1) \Big( (ad - bc - 2a + 2c)t - ad + 2a + b + d - 2 \Big) x^2 \\
&\quad - 81t^2(t-1)^2(at - ct - a + 1) \Big( (ad - bc - a + c)t - (d-1)(a+b-1) \Big) x.
\end{aligned}$$

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY  
*E-mail address*: kazu-utsumi@hiroshima-u.ac.jp